Wrocław University of Technology

DYNAMICS LECTURE 3

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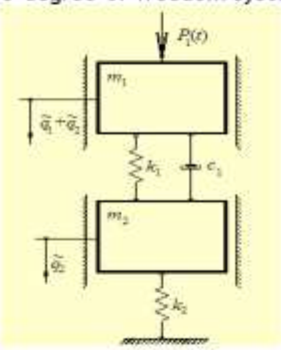
LECTURE 3

- Examples of calculating the stiffness matrices in geometrically determinate and indeterminate systems.
- Examples of forming an equation of motion of a discrete system: a beam supporting structure for a rotating motor.
- Examples of determining the mass matrix and the generalized vector of the exciting forces in discrete bar systems.

Illustrative Example 2DFS System with Inertial Coupling

two-degree-of-freedom system



$$\begin{aligned} E_k &= \frac{1}{2} m_1 (\dot{\tilde{q}}_1 + \dot{\tilde{q}}_2)^2 + \frac{1}{2} m_2 \dot{\tilde{q}}_2^{\,2} = \\ &= \frac{1}{2} m_1 \dot{\tilde{q}}_1^{\,2} + \frac{1}{2} 2m_1 \dot{\tilde{q}}_1 \dot{\tilde{q}}_2 + \frac{1}{2} m_1 \dot{\tilde{q}}_2^{\,2} + \frac{1}{2} m_2 \dot{\tilde{q}}_2^{\,2} = \\ &= \frac{1}{2} m_1 \dot{\tilde{q}}_1^{\,2} + \frac{1}{2} 2m_1 \dot{\tilde{q}}_1 \dot{\tilde{q}}_2 + \frac{1}{2} (m_1 + m_2) \dot{\tilde{q}}_2^{\,2} \end{aligned}$$

$$\Phi = \frac{1}{2} c_1 [(\dot{\tilde{q}}_1 + \dot{\tilde{q}}_2) - \dot{\tilde{q}}_2]^2 = \frac{1}{2} c_1 \dot{\tilde{q}}_1^{\,2}$$

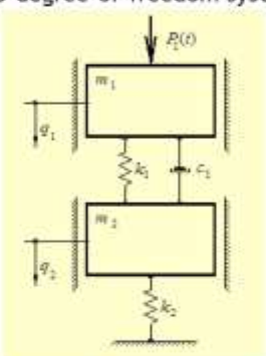
$$\begin{aligned} E_p &= \frac{1}{2} k_1 [(\tilde{q}_1 + \tilde{q}_2) - \tilde{q}_2]^2 + \frac{1}{2} k_2 (\tilde{q}_2)^2 = \\ &= \frac{1}{2} k_1 \tilde{q}_1^{\,2} + \frac{1}{2} k_2 \tilde{q}_2^{\,2} \end{aligned}$$

$$\tilde{\mathbf{q}} = \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{bmatrix} = \begin{bmatrix} q_1 - q_2 \\ q_2 \end{bmatrix}$$

$$\begin{bmatrix} m_1 & m_1 \\ m_1 & m_1 + m_2 \end{bmatrix} \begin{bmatrix} \ddot{\tilde{q}}_1 \\ \ddot{\tilde{q}}_2 \end{bmatrix} + \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{q}}_1 \\ \dot{\tilde{q}}_2 \end{bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_1 \end{bmatrix}$$

Illustrative Example 2DFS System with Static Coupling

two-degree-of-freedom system



$$E_k = \frac{1}{2} m_1 \dot{q}_1^{\,2} + \frac{1}{2} m_2 \dot{q}_2^{\,2}$$

$$\Phi = \frac{1}{2} c_1 (\dot{q}_1 - \dot{q}_2)^2 = \frac{1}{2} c_1 \dot{q}_1^{\,2} - \frac{1}{2} 2c_1 \dot{q}_1 \dot{q}_2 + \frac{1}{2} c_1 \dot{q}_2^{\,2}$$

$$\begin{aligned} E_p &= \frac{1}{2} k_1 (q_1 - q_2)^2 + \frac{1}{2} k_2 (q_2)^2 = \\ &= \frac{1}{2} k_1 q_1^2 - \frac{1}{2} 2k_1 q_1 q_2 + \frac{1}{2} k_1 q_2^2 + \frac{1}{2} k_2 q_2^2 = \\ &= \frac{1}{2} k_1 q_1^2 - \frac{1}{2} 2k_1 q_1 q_2 + \frac{1}{2} (k_1 + k_2) q_2^2 \end{aligned}$$

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} c_1 & -c_1 \\ -c_1 & c_1 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 + k_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ 0 \end{bmatrix}$$

System with Simultaneous Static and Inertial Coupling

- As the type of coupling depends on the choice of the generalized coordinates system, it is possible to choose such generalized coordinates that both static and inertial coupling will occur simultaneously.
- It seems also to be possible to find such a generalized coordinate system for which the equations of motion will be uncoupled.

System without Coupling (Decoupled System)

- The generalized coordinates system for which there is no coupling at all is called the principal generalized coordinates system.

Conclusions

Conclusions:

- the coupling of the equations of motion in MDOF systems is not a distinctive feature of the system but depends on the choice of the generalized coordinate system
- the MDOF system equations of motion can be coupled in three ways: inertially, elastically or inertially and elastically simultaneously
- uncoupled systems of equations, in which no coupling exists at all, are also possible

Stiffness Matrix in an Expanded Base of Coordinates

$$\hat{\mathbf{q}} = \begin{bmatrix} \mathbf{q} \\ \mathbf{x} \end{bmatrix}$$

subvector of dynamic degrees of freedom

$$\mathbf{q} = [q_1 \quad \dots \quad q_d]^T$$

subvector of static (geometric) degrees of freedom

$$\mathbf{x} = [x_1 \quad \dots \quad x_{n_{gd}}]^T$$

where Degree of Kinematic (Geometric) Indeterminacy in a Dynamic Sense is defined $n_{gd} = n_g - d$

The stiffness matrix in an expanded base of coordinates is defined as

$$\hat{\mathbf{K}} = \begin{bmatrix} \mathbf{K}_{qq} & \mathbf{K}_{qx} \\ \mathbf{K}_{xq} & \mathbf{K}_{xx} \end{bmatrix} \qquad \mathbf{K}_{qx} = \mathbf{K}_{xq}^T$$

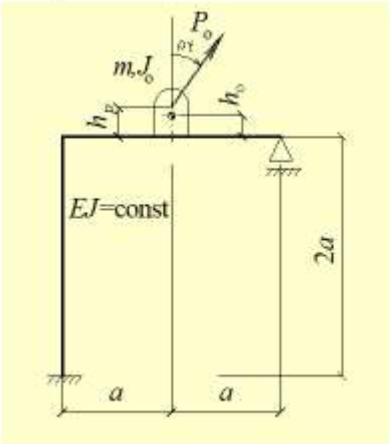
$$\dim \mathbf{K}_{qq} = d \times d \qquad \dim \mathbf{K}_{qx} = d \times n_{gd}$$

$$\dim \mathbf{K}_{xq} = n_{gd} \times d \qquad \dim \mathbf{K}_{xx} = n_{gd} \times n_{gd}$$

Illustrative Example

statically and kinematically indeterminate plane frame structure

Dynamic scheme of the frame



DATA

$a = 3\,\text{m}$	$m = 500\,kg$
$E = 200\,\text{GPa}$	$J_O = 20.8\,\text{kgm}^2$
$I = 9800\,\text{cm}^4$ (I 300)	$h_O = 0.25\,\text{m}$
$EI = \text{const}$	$h_P = 0.40\,\text{m}$
$EA = \infty$	$P_O = 1\,\text{kN}$
$GA = \infty$	$\omega = 30\,\text{rad/s}$

Number of degrees of freedom

$$d = d_\Delta + d_\varphi = 2 + 1 = 3$$

Static Condensation

The equilibrium conditions of the Displacement Method in the expanded base of coordinates has the form

$$\mathbf{K}_{xq}\,\mathbf{q} + \mathbf{K}_{xx}\,\mathbf{x} = \mathbf{0}$$ from here $$\mathbf{x} = -\mathbf{K}_{xx}^{-1}\mathbf{K}_{xq}\mathbf{q}$$

From the identity

$$\mathbf{K}_{qq}\mathbf{q} + \mathbf{K}_{qx}\mathbf{x} = \mathbf{K}\mathbf{q}$$

after substituting **x**, one can achieve the stiffness matrix in the base of generalized coordinates from formula

$$\mathbf{K} = \mathbf{K}_{qq} - \mathbf{K}_{qx}\mathbf{K}_{xx}^{-1}\mathbf{K}_{xq}$$

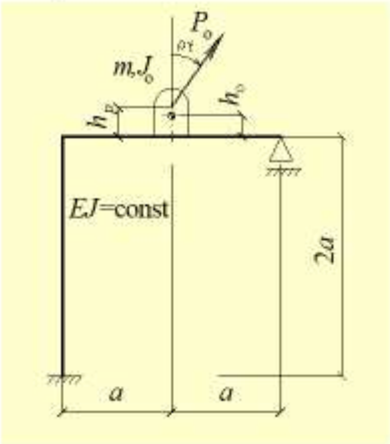
For an SDOF system the kinematic indeterminacy degree $n_{gd} = 1$ from one can find the equivalent stiffness coefficient

$$k = k_{qq} - \frac{k_{qx}k_{xq}}{k_{xx}}$$

Illustrative Example

statically and kinematically indeterminate plane frame structure

Dynamic scheme of the frame

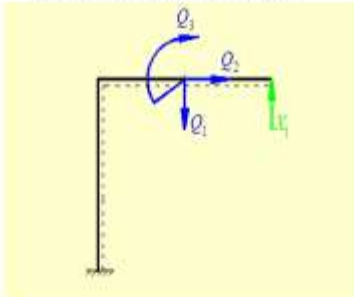


Force Method

The degree of static indeterminacy of a system (number of hyperstatics)

$$n_h = e - 3t = 4 - 3 \cdot 1 = 1$$

Scheme of coordinates.



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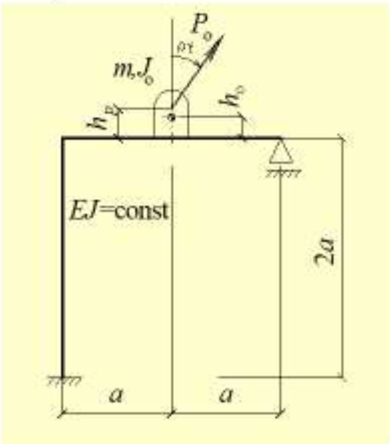
Illustrative Example

statically and kinematically indeterminate plane frame structure

Dynamic scheme of the frame



Displacement Method

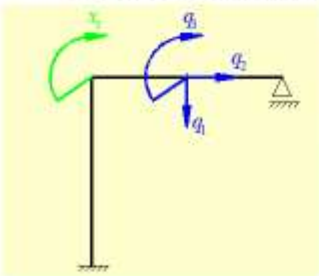
The degree of static geometric indeterminacy

$$n_g = n_\Delta + n_\varphi = 2 + 2 = 4$$

The degree of static geometric indeterminacy in a dynamic sense

$$n_{gd} = n_g - d = 4 - 3 = 1$$

Scheme of coordinates



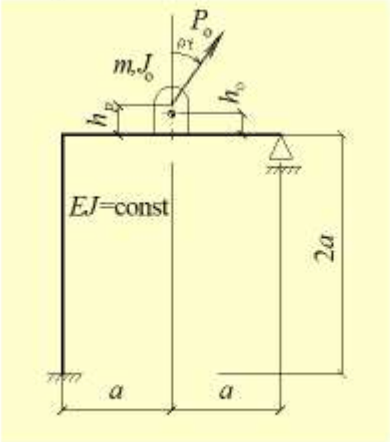
$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$\mathbf{x} = [x_1]$$

Illustrative Example

statically and kinematically indeterminate plane frame structure

Dynamic scheme of the frame



Displacement Method

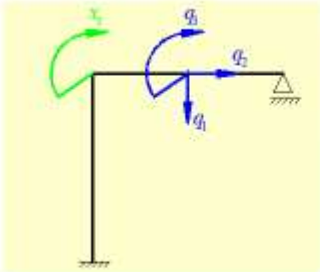
The degree of static geometric indeterminacy

$$n_g = n_\alpha + n_\varphi = 2 + 2 = 4$$

The degree of static geometric indeterminacy in a dynamic sense

$$n_{gd} = n_g - d = 4 - 3 = 1$$

Scheme of coordinates



$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$\mathbf{x} = [x_1]$$

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The Displacement Method

The expanded base of kinematic coordinates is defined as the vector

$$\hat{\mathbf{q}} = \begin{bmatrix} \mathbf{q} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ x_1 \end{bmatrix}$$

The stiffness matrix in an expanded base of coordinates is defined as

$$\hat{\mathbf{K}} = \begin{bmatrix} \mathbf{K}_{qq} & \mathbf{K}_{qx} \\ \mathbf{K}_{xq} & \mathbf{K}_{xx} \end{bmatrix} = \begin{bmatrix} \frac{15EI}{l^3} & 0 & \frac{-3EI}{l^2} & \frac{-6EI}{l^2} \\ 0 & \frac{3EI}{2l^3} & 0 & \frac{-3EI}{2l^2} \\ \frac{-3EI}{l^2} & 0 & \frac{7EI}{l} & \frac{2EI}{l} \\ \frac{-6EI}{l^2} & \frac{-3EI}{2l^2} & \frac{2EI}{l} & \frac{6EI}{l} \end{bmatrix}$$

$$\mathbf{K}_{qx} = \mathbf{K}_{xq}^T = \begin{bmatrix} \frac{-6EI}{l^2} \\ \frac{-3EI}{2l^2} \end{bmatrix}$$

$$\mathbf{K}_{xx} = \left[\frac{6EI}{l^2} \right]$$

Stiffnes matrix

Static Condensation yields the stiffness matrix in generalized coordinates base

$$\mathbf{K}=\mathbf{K}_{qq}-\mathbf{K}_{qx}\mathbf{K}_{xx}^{-1}\mathbf{K}_{xq}=$$

$$=\begin{bmatrix} \frac{6EI}{l^3} & -\frac{3EI}{2l^3} & -\frac{EI}{l^2} \\ -\frac{3EI}{2l^3} & \frac{9EI}{8l^3} & \frac{EI}{2l^2} \\ -\frac{EI}{l^2} & \frac{EI}{2l^2} & \frac{19EI}{3l} \end{bmatrix}=\begin{bmatrix} 6.53\cdot10^6 & -1.09\cdot10^6 & -2.18\cdot10^6 \\ -1.09\cdot10^6 & 0.82\cdot10^6 & 1.09\cdot10^6 \\ -2.18\cdot10^6 & 1.09\cdot10^6 & 41.14\cdot10^6 \end{bmatrix}$$

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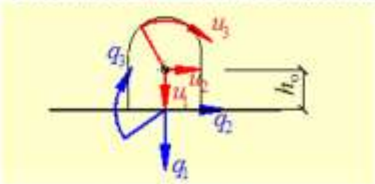
Mass matrix

Generalized and local coordinates associated with mass center



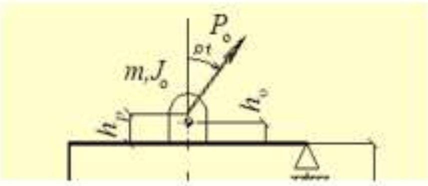
$$\{\mathbf{m}\}=diag(m \quad m \quad J_O)=\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J_O \end{bmatrix}$$

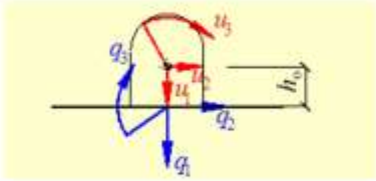
$$\mathbf{q}=\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$\mathbf{u}=\mathbf{A}_m\mathbf{q} \quad \rightarrow \quad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & h_O \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

$$\mathbf{B}=\mathbf{A}_m^T\cdot\{\mathbf{m}\}\cdot\mathbf{A}_m=\begin{bmatrix} m & 0 & 0 \\ 0 & m & m\,h_O \\ 0 & m\,h_O & J_O+m\,h_O^2 \end{bmatrix}=\begin{bmatrix} 500 & 0 & 0 \\ 0 & 500 & 125 \\ 0 & 125 & 52.08 \end{bmatrix}$$

Generalized Forces Vector





$$h_p \to h_o$$

$$\mathbf{F} = \begin{bmatrix} -P_O \cos pt \\ P_O \sin pt \\ P_O\, h_p \sin pt \end{bmatrix} = \begin{bmatrix} 0 \\ P_O \\ P_O\, h_p \end{bmatrix} \sin pt + \begin{bmatrix} -P_0 \\ 0 \\ 0 \end{bmatrix} \cos pt = \mathbf{F}_S \sin pt + \mathbf{F}_C \cos pt$$

Matrix Equation of Motion

$$\mathbf{B}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}(t)$$

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & m h_O \\ 0 & m h_O & J_O + m h_O^2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{bmatrix} + \begin{bmatrix} 6EI/l^3 & -3EI/2l^3 & -EI/l^2 \\ -3EI/2l^3 & 9EI/8l^3 & EI/2l^2 \\ -EI/l^2 & EI/2l^2 & 19EI/3l \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ P_O \\ P_O\, h_p \end{bmatrix} \sin pt + \begin{bmatrix} -P_0 \\ 0 \\ 0 \end{bmatrix} \cos pt$$