Wrocław University of Technology

DYNAMICS LECTURE 4

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LECTURE 4

- The eigenproblem of a discrete system.
- Free vibration of the discrete system.
- Damping in civil engineering structures.
- Harmonically excited steady-state vibration in discrete systems (direct method).
- The Orthogonality Principle of natural vibration, the modal transformation method.
- Harmonic excitation in a one-degree-of-freedom system.
- The use of the modal transformation method for analysing harmonically excited steady-state vibration in multi-degree-of-freedom systems.

Eigenproblem of a discrete system

Matrix equations of motion in the generalized coordinate base

$$\mathbf{B}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}(t)$$

- $\mathbf{B}$ mass (inertia) matrix of a system
- $\mathbf{C}$ damping matrix of a system
- $\mathbf{K}$ stiffness matrix of a system
- $\mathbf{F}(t)$ vector of external generalized forces acting on a system
- $\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}$ generalized coordinates, velocities and accelerations vectors respectively

Eigenproblem of a discrete system

The problem of free vibrations requires that the force vector be equal to zero in equation of motion.
If the system is also undamped, the equation of motion can be written in form

$$\mathbf{B}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$$

For the free vibrations of the undamped structure, one can guess (Lucky Guess Method) the form of the solutions of this equation of motion

$$\mathbf{q}(t) = \mathbf{a}\sin(\omega t + \varphi)$$

Two times differentiation of this expression with respect to time leads to formula

$$\ddot{\mathbf{q}} = -\omega^2\mathbf{q}$$

The substitution of these expressions to equation of motion gives

$$(\mathbf{K} - \omega^2\mathbf{B})\mathbf{q} = \mathbf{0}$$

Eigenproblem of a discrete system

For any one natural frequency ω_{ni} (eigenvalue of $\lambda_i=\omega_{ni}^2$) such a solution exists

$$\mathbf{q} = \mathbf{w}_i$$

That

$$(\mathbf{K} - \omega_{ni}^{2}\mathbf{B})\mathbf{w} = 0$$

The vector $\mathbf{w}_i$ is called an eigenvector (i-th normal or natural mode of vibration or modal shape). The eigenvector coordinates are generalized displacements, which describe the modal shape, that is they specify how, for each natural angular frequency ω_{ni} , the various elements of the system move in relation to each other.

It is easy to prove that each non-zero column of the adjugate (adjoint) matrix

$$\mathrm{adj}\mathbf{A}_i = \mathrm{adj}(\mathbf{K} - \omega_i^2\mathbf{B})$$

is an eigenvector

Eigenproblem of a discrete system

Proof:

Let us consider the formula for the inverse matrix $\mathbf{A}_i^{-1} = \dfrac{\mathrm{adj}\,\mathbf{A}_i}{\det \mathbf{A}_i}$

This formula can be written in another form, i.e. $\mathbf{A}_i^{-1} \det \mathbf{A}_i = \mathrm{adj}\,\mathbf{A}_i$

Premultiplication of this equation by matrix $\mathbf{A}$ and postmultiplication by any non-zero vector $\mathbf{v}$ results in

$$\det \mathbf{A}_i \cdot \mathbf{v} = \mathbf{A}_i\, \mathrm{adj}\mathbf{A}_i \cdot \mathbf{v}$$

Since $\det \mathbf{A}_i = 0$ and designates a new vector $\mathbf{b} = \mathrm{adj}\mathbf{A}_i \cdot \mathbf{v}$, this equation can be written in the form

$$\det \mathbf{A}_i\, \mathbf{v} = \mathbf{A}_i\, (\mathrm{adj}\mathbf{A}_i \cdot \mathbf{v}) = \mathbf{A}_i\mathbf{b} = (\mathbf{K} - \omega_{ni}^2\mathbf{B})\mathbf{b} = \mathbf{0}$$

Eigenproblem of a discrete system

As v could be any vector it could also be a versor, with 1 on *j*-th position. Then the multiplication $\mathbf{A}_i\mathbf{v}$ takes out *j*-th column from the adjoint matrix. Comparing equation

$$\det \mathbf{A}_i \mathbf{v} = \mathbf{A}_i (\mathrm{adj}\mathbf{A}_i \cdot \mathbf{v}) = \mathbf{A}_i\mathbf{b} = (\mathbf{K} - \omega_{m}^{2}\mathbf{B})\mathbf{b} = \mathbf{0}$$

and

$$(\mathbf{K} - \omega^2\mathbf{B})\mathbf{q} = \mathbf{0}$$

one can write down

$$\mathbf{w}_i = \mathbf{b} = \mathrm{adj}(\mathbf{K} - \omega_{m}^{2}\mathbf{B})$$

The eigenvectors can be normalized

$$\mathbf{w}_{i,\mathrm{norm}} = \mathbf{w}_i / N_i$$

usually using the norms

$$\mathbf{w}_{i,\mathrm{norm}} = \mathbf{w}_i / N_i \qquad N_i = \|\mathbf{w}_i\| = \max_j |w_{ji}| \qquad \text{or} \qquad N_i = \|\mathbf{w}_i\| = \sqrt{\mathbf{w}_i^T \mathbf{B}\, \mathbf{w}_i}$$

Modal and Spectral Matrices

For each value of eigenfrequency ω_m (natural angular frequency) satisfying the characteristic equation one may solve equation $(\mathbf{K} - \omega_{m}^{2}\mathbf{B})\mathbf{w}_i = \mathbf{0}$. This solution, with an accuracy up to a constant multiplier (multiplicative constant), is the eigenvector

$$\mathbf{w}_i = \begin{bmatrix} q_{i1} \\ q_{i2} \\ \vdots \\ q_{in} \end{bmatrix} = \begin{bmatrix} w_{i1} \\ w_{i2} \\ \vdots \\ w_{in} \end{bmatrix}$$

This solutions describe the normal modes (shapes) which may be conveniently arranged in the columns of a matrix known as the modal matrix , that is

$$\mathbf{W} = [\mathbf{w}_1 \quad \mathbf{w}_2 \quad \cdots \quad \mathbf{w}_n] = \begin{bmatrix} w_{11} & w_{12} & \cdots & w_{1n} \\ w_{21} & w_{22} & \cdots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{bmatrix}$$

Modal and Spectral Matrices

The n eigenvalues $\lambda_i = \omega_{ni}^2$ can be assembled into a diagonal matrix $\mathbf{\Omega}^2 = \{\boldsymbol{\omega}^2\}$ which is known as a spectral matrix of the eigenproblem, that is

$$\mathbf{\Omega}^2 = \{\boldsymbol{\omega}^2\} = \begin{bmatrix} \omega_{n1}^2 & 0 & \dots & 0 \\ 0 & \omega_{n2}^2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \omega_{nn}^2 \end{bmatrix}$$

By using the modal and spectral matrices it is possible to assemble all of these relations into a single matrix equation

$$\mathbf{K}\,\mathbf{W} = \mathbf{B}\mathbf{W}\mathbf{\Omega}^2$$

Normal and Natural Mode of Vibration

- When a system is capable of vibrating with more than one frequency, but is actually vibrating freely at only one of its possible normal frequencies, the system is said to be vibrating in one of its "normal modes."
- A normal mode of vibration is a mode of vibration that is uncoupled from (i.e., can exist independently of) other modes of vibration of a system.
- When vibration of the system is defined as an eigenvalue problem, the normal modes are the eigenvectors and the normal mode frequencies are the eigenvalues.
- The natural (normal) mode of vibration is a mode of vibration assumed by a system when vibrating freely.
- The mode of vibration associated with the lowest natural frequency of a system is referred to as the first (basic) mode. The next higher frequency is the second, and so on.

Orthogonality of Normal Modes (Orthogonality Principle)

- The most important property of the normal modes is their orthogonality. For this reason the normal modes can be used to uncouple the matrix equations of motion.
- The solution of a set of separate differential equations is significantly easier than the solution of a set of coupled differential equations.
- Premultiplication of Eq. $\mathbf{K W} = \mathbf{B W \Omega}^2$ by matrix $\mathbf{W}^T$ yields equation

$$\mathbf{W}^T\mathbf{K W} = \mathbf{W}^T\mathbf{B W \Omega}^2$$

- After transposition of this Eq., and taking into account the symmetry of matrices **B** and **K**, this Eq. can be written in form

$$\mathbf{W}^T\mathbf{K W} = \mathbf{\Omega}^2\mathbf{W}^T\mathbf{B W}$$

- The left sides of these Eqs.are the same, thus the right sides must also be the same, that is

Orthogonality of Normal Modes (Orthogonality Principle)

$$\mathbf{W}^T\mathbf{B W \Omega}^2 = \mathbf{\Omega}^2\mathbf{W}^T\mathbf{B W}$$

In general, this Eq. is true only if matrix $\mathbf{W}^T\mathbf{B W}$ is a diagonal matrix. That matrix is called the principal masses matrix and its elements - modal masses

$$\mathbf{W}^T\mathbf{B W} = \{\mathbf{m}_o\}$$

Substituting this Eq. into Eq. $\mathbf{W}^T\mathbf{K W} = \mathbf{\Omega}^2\mathbf{W}^T\mathbf{B W}$ implies that matrix $\mathbf{W}^T\mathbf{K W}$ must also be a diagonal matrix (principal stiffnesses matrix) and its elements - modal stiffnesses

$$\mathbf{W}^T\mathbf{K W} = \{\mathbf{k}_o\} = \{\mathbf{m}_o\}\{\omega^2\}$$

Orthogonality of Normal Modes (Orthogonality Principle)

If the flexibility matrix is used to formulate the equation of motion, the reduced equation of motion has the form

$$\mathbf{DB}\ddot{\mathbf{q}} + \mathbf{q} = \mathbf{0}$$

where **D** is a flexibility matrix. Now eigenproblem can be formulated

$$(\mathbf{DB} - \omega^{-2}\mathbf{I})\mathbf{q} = \mathbf{0}$$

And consequently

$$\det(\mathbf{DB} - \omega^{-2}\mathbf{I}) = 0$$

Orthogonality of Normal Modes (Orthogonality Principle)

Conclusions:

- It must be noticed that eigenvectors are orthogonal with respect to both the mass and the stiffness matrix, but eigenvectors **are not orthogonal** with respect to the flexibility matrix.
- Each eigenvector is determined in terms of an arbitrary constant and can normalized arbitrarily.
- If *n*>*d* (base of generalized coordinates is not minimal) the eigenfrequency $\omega = \infty$ can appear. These solutions must be neglected.

Natural Vibration
Ambiguity of Term "Natural Vibration"

The term "natural vibration" is somewhat problematic due to the conventions of Polish terminology. Following Langer [5], the Polish term "*drgania własne*" (the lexical equivalent of the English term "natural vibration") does not relate to a physical phenomenon. It does not designate vibration, but a mathematical form of the general solution (total integral) of an inhomogeneous differential equation of motion without damping, Eq. (3.47), which describes a predisposition of the structure to vibrate freely with accordance to natural frequencies and natural (normal) forms of vibration.

In English terminology, the term "natural vibration" appears predominantly in connection to such terms as "frequency of natural vibration" or "mode of natural vibration". This indicates that "natural vibration" are identified with "free vibration". As Harris writes, "[t]he natural mode of vibration is a mode of vibration assumed by a system when vibrating freely.", [1]. Thus, the expression, as used in the English terminology, clearly describes a physical phenomenon.

Therefore, the term "natural vibration" may be ambiguous, as it will have one meaning when used in the sense attached to it in English terminology, and a different one if used as a translation of the Polish "*drgania własne*". It is suggested that the term "natural vibration" should be used uniformly in the meaning equivalent to free vibration to avoid this ambiguity.

Free Vibration
Undamped Free Vibration

- Free vibration (natural vibration) is a physical phenomenon which occurs in an undamped MDOF system when there is no excitation and the initial conditions are given: $\mathbf{q}(0)=\mathbf{q}_o$ and $\dot{\mathbf{q}}(0)=\dot{\mathbf{q}}_o$.
- If the base of generalized coordinates is minimal (*n*=*d*) the solution

$$\mathbf{q}(t)=\sum_{i=1}^{d}\mathbf{w}_i(s_i\sin\omega_{ni}t+c_i\cos\omega_{ni}t)=\mathbf{W}\cdot\{\sin\boldsymbol{\omega}_n\mathbf{t}\}\cdot\mathbf{s}+\mathbf{W}\cdot\{\cos\boldsymbol{\omega}_n\mathbf{t}\}\cdot\mathbf{c}$$

may be used to determinate the free vibration of the system where ,

$$\begin{aligned}\{\sin\boldsymbol{\omega}_n\mathbf{t}\}&=\mathrm{diag}\,(\sin\omega_{n1}t\quad\sin\omega_{n2}t\quad\dots\quad\sin\omega_{nd}t)\\\{\cos\boldsymbol{\omega}_n\mathbf{t}\}&=\mathrm{diag}\,(\cos\omega_{n1}t\quad\cos\omega_{n2}t\quad\dots\quad\cos\omega_{nd}t)\\\mathbf{s}=[s_1\quad s_2\quad\dots\quad s_d]^T&\quad\mathbf{c}=[c_1\quad c_2\quad\dots\quad c_d]^T\end{aligned}$$

and

$$\begin{aligned}\mathbf{q}_o=\mathbf{W}\mathbf{c}\quad&\rightarrow\quad\mathbf{c}=\mathbf{W}^{-1}\mathbf{q}_o\\\dot{\mathbf{q}}_o=\mathbf{W}\cdot\{\boldsymbol{\omega}\}\cdot\mathbf{s}\quad&\rightarrow\quad\mathbf{s}=\{\boldsymbol{\omega}^{-1}\}\cdot\mathbf{W}^{-1}\dot{\mathbf{q}}_o\end{aligned}$$

Finally

$$\mathbf{q}(t)=\mathbf{W}\cdot\{\cos\boldsymbol{\omega}_n\mathbf{t}\}\cdot\mathbf{W}^{-1}\mathbf{q}_o+\mathbf{W}\cdot\left\{\frac{\sin\boldsymbol{\omega}_n\mathbf{t}}{\boldsymbol{\omega}_n}\right\}\cdot\mathbf{W}^{-1}\dot{\mathbf{q}}_o$$

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Damped Free Vibration

The equation of motion which describes damped free vibration has the form

$$\mathbf{B}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$$

with initial conditions $\mathbf{q}(0) = \mathbf{q}_o$ and $\dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_o$

Usually it is convenient to assume that the damping matrix is proportional to either the mass or the stiffness matrix, but the best assumption is that the damping matrix is proportional to both of them (Rayleigh damping),

$$\mathbf{C} = \kappa\mathbf{K} + \mu\mathbf{B}$$

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Damped Free Vibration Modal Transformation Method

The equation of motion which describes damped free vibration has the form

$$\mathbf{B}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$$

with initial conditions $\mathbf{q}(0) = \mathbf{q}_o$ and $\dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_o$

Usually it is convenient to assume that the damping matrix is proportional to either the mass or the stiffness matrix, but the best assumption is that the damping matrix is proportional to both of them (Rayleigh damping),

$$\mathbf{C} = \kappa\mathbf{K} + \mu\mathbf{B}$$

where μ is the dimensional damping parameter and κ is also a dimensional parameter called the retardation time.

Damped Free Vibration Modal Transformation Method

The transformation from the new set of coordinates to the generalized coordinates , such as

$$\mathbf{q} = \mathbf{W}\mathbf{r}$$

is substituted into Eq.

By premultiplying the equation by the transposed modal matrix $\mathbf{W}_i^T$ and making use of the orthogonal properties of the modal matrix (eigenvectors), the matrix equation has the form

$$\{\mathbf{m}_o\}\ddot{\mathbf{r}} + \{\mathbf{c}_o\}\dot{\mathbf{r}} + \{\mathbf{k}_o\}\mathbf{r} = \mathbf{0}$$

where

$$\{\mathbf{c}_o\} = \mathbf{W}^T\mathbf{C}\mathbf{W}$$

and

$$\mathbf{q}_o = \mathbf{W}\mathbf{r}_o \qquad \dot{\mathbf{q}}_o = \mathbf{W}\dot{\mathbf{r}}_o$$

Eq. is the matrix form of a set of uncoupled equations.

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Damped Free Vibration Principal Coordinates System

- The coordinates by which it is possible to uncouple the MDOF system, described by vector **r**, are called the principal coordinates system.
- The principal coordinates vector can be achieved from the generalized coordinates vector with the use of modal matrix $\mathbf{W}$ transformation $(\mathbf{q} = \mathbf{W}\mathbf{r})$.
- The solution in the base of normal coordinates (in matrix notation) has the form

$$\mathbf{r}(t) = \left\{ e^{-\zeta\omega_n t} \frac{\cos(\omega_d t - \beta)}{\cos\beta} \right\} \cdot \mathbf{r}_o + \left\{ e^{-\zeta\omega_n t} \frac{\sin\omega_d t}{\omega_n \cos\beta} \right\} \cdot \dot{\mathbf{r}}_o$$

where

$$\left\{ e^{-\zeta\omega_n t} \frac{\cos(\omega_d t - \beta)}{\cos\beta} \right\} = \mathrm{diag}\left(e^{-\zeta_j\omega_{nj} t} \frac{\cos(\omega_{dj} t - \beta_i)}{\cos\beta_i} \right)$$

$$\left\{ e^{-\zeta\omega_n t} \frac{\sin\omega_d t}{\omega_n \cos\beta} \right\} = \mathrm{diag}\left(e^{-\zeta_j\omega_{nj} t} \frac{\sin\omega_{dj} t}{\omega_{nj} \cos\beta_i} \right)$$

Damped Free Vibration
Principal Coordinates System

after retransformation

$$\mathbf{q}(t) = \mathbf{W} \cdot \left\{ e^{-\alpha\omega_n t} \frac{\cos(\omega_d t - \beta)}{\cos\beta} \right\} \cdot \mathbf{W}^{-1}\mathbf{q}_o + \mathbf{W} \cdot \left\{ e^{-\alpha\omega_n t} \frac{\sin\omega_d t}{\omega_n \cos\beta} \right\} \cdot \mathbf{W}^{-1}\dot{\mathbf{q}}_o$$

The inverse modal matrix can be calculated without a formal inverse procedure, since from the Orthogonality Principle results

$$\mathbf{W}^{-1} = \{\mathbf{m}_o\}^{-1}\mathbf{W}^T\mathbf{B} = \{\mathbf{k}_o\}^{-1}\mathbf{W}^T\mathbf{K}$$

Harmonic Forced Vibration
Direct Method

Let us assume that the force excitation vector is in the form

$$\mathbf{F}(t) = \mathbf{F}_S \sin \omega t + \mathbf{F}_C \cos \omega t$$

In equation of motion

$$\mathbf{B}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}(t)$$

The steady-state response of equation of motion could be looked for (Lucky Guess Method or method of undetermined coefficients) also in harmonic form

$$\mathbf{q}(t) = \mathbf{q}_S \sin \omega t + \mathbf{q}_C \cos \omega t$$

By substituting formulas into Equation of motion and then comparing the terms at the sinusoidal and cosinusoidal components of the solution respectively, the algebraic set of equations is achieved

Harmonic Forced Vibration Direct Method

$$(\mathbf{K}-\omega^2\mathbf{B})\mathbf{q}_S-\omega\mathbf{C}\mathbf{q}_C=\mathbf{F}_S$$
$$\omega\mathbf{C}\mathbf{q}_S+(\mathbf{K}-\omega^2\mathbf{B})\mathbf{q}_C=\mathbf{F}_C$$

$$\begin{bmatrix}\mathbf{K}-\omega^2\mathbf{B} & -\omega\mathbf{C}\\ \omega\mathbf{C} & \mathbf{K}-\omega^2\mathbf{B}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{q}_S\\ \mathbf{q}_C\end{bmatrix}=\begin{bmatrix}\mathbf{F}_S\\ \mathbf{F}_C\end{bmatrix}$$

If the influence of damping is negligible, Eq. is reduced to the simple matrix form

$$(\mathbf{K}-\omega^2\mathbf{B})\,\mathbf{q}_{S,C}=\mathbf{F}_{S,C}$$

which is valid for both the sinusoidal $\mathbf{q}_s$ and the cosinusoidal $\mathbf{q}_c$ component of solution

Harmonic Forced Vibration Direct Method

Applying a procedure analogous to the one described previously, the set of algebraic equations has the form

$$\begin{bmatrix}\mathbf{I}-\omega^2\mathbf{DB} & -\omega\mathbf{DC}\\ \omega\mathbf{DC} & \mathbf{I}-\omega^2\mathbf{DB}\end{bmatrix}\cdot\begin{bmatrix}\mathbf{q}_S\\ \mathbf{q}_C\end{bmatrix}=\begin{bmatrix}\mathbf{DF}_S\\ \mathbf{DF}_C\end{bmatrix}$$

If damping matrix is **C**=0, this equation is simplified to the form

$$(\mathbf{I}-\omega^2\mathbf{DB})\mathbf{q}_{S,C}=\mathbf{DF}_{S,C}$$

which is valid for both the sinusoidal $\mathbf{q}_s$ and the cosinusoidal $\mathbf{q}_c$ component of solution

Harmonic Forced Vibration
Direct Method

Conclusions

- **Advantages of the Direct Method**

in comparison to the Modal Transformation Method approach:

- There is no need to solve the eigenproblem to achieve the steady-state response of the system.
- There is no need to assume that the damping matrix is proportional to either the mass or the stiffness matrix or to both of them to achieve the steady-state response of the system.

- **Disadvantages of the Direct Method**

in comparison to Modal Transformation Method approach:

- It is necessary to solve a doubled set of coupled algebraic equations.
- It is impossible to reduce the base of coordinates used to the determine the solution (the dynamic condensation cannot be performed, see Chapter 3.18)

Harmonic Forced Vibration
Modal Transformation Method

After substituting $\mathbf{q} = \mathbf{W}\mathbf{r}$ into Eq. Of motion and then premultiplicating by $\mathbf{W}^T$, the equation of motion has the form

$$\mathbf{W}^T\mathbf{B}\,\mathbf{W}\ddot{\mathbf{r}} + \mathbf{W}^T\mathbf{C}\,\mathbf{W}\dot{\mathbf{r}} + \mathbf{W}^T\mathbf{K}\,\mathbf{W}\mathbf{r} = \mathbf{W}^T\mathbf{F} = \mathbf{R}(t)$$

Using the Orthogonality Principle and additionally

$$\mathbf{W}^T\mathbf{C}\,\mathbf{W} = \{2\alpha\sqrt{\mathbf{k}_o\mathbf{m}_o}\} = \mathrm{diag}(2\alpha_i\sqrt{k_{o,i}m_{o,i}})$$

the equation of motion takes the diagonal form

$$\{\mathbf{m}_o\}\ddot{\mathbf{r}} + \{2\alpha\sqrt{\mathbf{k}_o\mathbf{m}_o}\}\dot{\mathbf{r}} + \{\mathbf{k}_o\}\mathbf{r} = \mathbf{R}(t)$$

Harmonic Forced Vibration
Modal Transformation Method

Considering relations and one can achieve

$$\mathbf{q}_S = \mathbf{W}\{\mathbf{h}_1\}\mathbf{W}^T\mathbf{F}_S + \mathbf{W}\{\mathbf{h}_2\}\mathbf{W}^T\mathbf{F}_c = \mathbf{H}_1\mathbf{F}_S + \mathbf{H}_2\mathbf{F}_C$$
$$\mathbf{q}_C = \mathbf{W}\{\mathbf{h}_1\}\mathbf{W}^T\mathbf{F}_C - \mathbf{W}\{\mathbf{h}_2\}\mathbf{W}^T\mathbf{F}_S = \mathbf{H}_1\mathbf{F}_C - \mathbf{H}_2\mathbf{F}_S$$

or in matrix block form

$$\begin{bmatrix}\mathbf{q}_S\\ \mathbf{q}_C\end{bmatrix} = \begin{bmatrix}\mathbf{H}_1 & \mathbf{H}_2\\ -\mathbf{H}_2 & \mathbf{H}_1\end{bmatrix}\begin{bmatrix}\mathbf{F}_S\\ \mathbf{F}_C\end{bmatrix}$$

where

$$\mathbf{H}_1 = \mathbf{W}\{\mathbf{h}_1\}\mathbf{W}^T$$
$$\mathbf{H}_2 = \mathbf{W}\{\mathbf{h}_2\}\mathbf{W}^T$$

If damping influence is negligible $\mathbf{q} = \mathbf{H}_1\mathbf{F}$

which is valid for both the sinusoidal and the cosinusoidal component of solution, and $h_i = \frac{1}{k_{oi}(1-\eta_i^2)}$

Harmonic Forced Vibration
Modal Transformation Method

In the particular situation of a harmonic excitation

$$\mathbf{R}(t) = \mathbf{W}^T(\mathbf{F}_S \sin\omega t + \mathbf{F}_C \cos\omega t) = \mathbf{R}_S \sin\omega t + \mathbf{R}_C \cos\omega t)$$

and

$$\mathbf{r}(t) = \mathbf{r}_S \sin\omega t + \mathbf{r}_C \cos\omega t$$

iIt is possible to write

$$\mathbf{r}_C = \{\mathbf{h}_1\}\mathbf{R}_S + \{\mathbf{h}_2\}\mathbf{R}_C,$$
$$\mathbf{r}_S = \{\mathbf{h}_1\}\mathbf{R}_C - \{\mathbf{h}_2\}\mathbf{R}_S,$$

where

$$\{\mathbf{h}_1\} = \mathrm{diag}(h_{11} \quad h_{12} \quad \ldots \quad h_{1d}) \qquad \{\mathbf{h}_2\} = \mathrm{diag}(h_{21} \quad h_{22} \quad \ldots \quad h_{2d})$$
$$h_{1i} = \frac{1}{k_{oi}}\frac{1-\eta_i^2}{(1-\eta_i^2)^2 + (2\alpha_i\eta_i)^2} \qquad h_{2i} = \frac{1}{k_{oi}}\frac{2\alpha_i\eta_i}{(1-\eta_i^2)^2 + (2\alpha_i\eta_i)^2}$$
$$\eta_i = \omega/\omega_{oi}$$

nominator/denominator

Harmonic Forced Vibration
Modal Transformation Method

- **Conclusions**
- **Advantages of the Modal Transformation Method** in comparison to the Direct Method approach:
- It is possible to use this approach for another type of forcing excitations, i.e. not for harmonic excitation only.
- The whole analysis can be conducted with the use of SDOF systems only
- It is possible to arbitrarily specify the damping ratios for each mode.
- **Disadvantages of the Modal Transformation Method** in comparison to the Direct Method approach:
- The eigenproblem analysis must be accomplished
- The assumption is necessary that the damping matrix is proportional to either the mass or the stiffness matrix or to both of them to achieve the steady-state response of the system.