Wrocław University of Technology

DYNAMICS LECTURE 2

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LECTURE 2

- Second order Lagrange's equations.
- Systems of coordinates and their transformations.
- The energetic balance and the matrix equation of motion of a discrete system.
- Elastic bonds in discrete bar systems, the definition of the displacement and stiffness matrices.
- Examples of calculating the displacement matrix in statically determinate and indeterminate systems.

Lagrangian Equations

In MDOF systems formulating a differential equation of motion for a vibrating system can be achieved in terms of the energies of the system with the use of the Lagrangian equations

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}_i} - \frac{\partial E_k}{\partial q_i} + \frac{\partial \Phi}{\partial \dot{q}_i} + \frac{\partial E_p}{\partial q_i} = \frac{\partial W}{\partial q_i}, \quad i = 1,2,\ldots,n$$

or in vector notation

$$\frac{d}{dt} grad\, E_k(\dot{\mathbf{q}}) - grad\, E_k(\mathbf{q}) + grad\, \Phi(\dot{\mathbf{q}}) + grad\, E_p(\mathbf{q}) = grad\, W(\mathbf{q})$$

where

E_k - total kinetic energy of the system
E_p - total potential energy of the system
Φ - Rayleigh Dissipation Function
W - virtual work of external generalized forces
q_i , $\dot{q}_i$ - generalized coordinate (displacement) and velocity
$\mathbf{q}$, $\dot{\mathbf{q}}$ - generalized coordinates and velocity vectors

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Lagrangian Equations

In the case of small vibration around the equilibrium point the kinetic energy in general does not depend on generalized displacements q_i. Then

$$\frac{\partial E_k}{\partial q_i} = 0 \qquad grad\, E_k(\mathbf{q}) = \mathbf{0}$$

and the Lagrangian equations take the form

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}_i} + \frac{\partial \Phi}{\partial \dot{q}_i} + \frac{\partial E_p}{\partial q_i} = \frac{\partial W}{\partial q_i}, \quad i = 1,2,\ldots,n$$

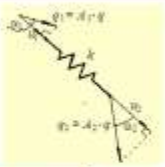
or in vector notation

$$\frac{d}{dt} grad\, E_k(\dot{\mathbf{q}}) + grad\, \Phi(\dot{\mathbf{q}}) + grad\, E_p(\mathbf{q}) = grad\, W(\mathbf{q})$$

The potential energy

The potential energy

$$E_p = \frac{1}{2} k (u_2 - u_1)^2$$



Stiffness Matrix in Local Coordinates Base of Spring Elements

$$E_p = \frac{1}{2}\sum_j k_j u_j^2 = \frac{1}{2}\mathbf{u}^T\{\mathbf{k}\}\mathbf{u}$$

$$\{\mathbf{k}\} = diag(k_1 \quad k_2 \quad k_3 \quad \ldots) = \begin{bmatrix} k_1 & 0 & 0 & \cdots \\ 0 & k_2 & 0 & \cdots \\ 0 & 0 & k_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Transformation from Generalized to Local Coordinates

$$\mathbf{u} = \mathbf{A}_k\mathbf{q}$$

$\mathbf{A}_k$ is the transformation matrix whose elements depend on the configuration of the structure only; the dimension of this transformation matrix is (number of local coordinates of spring elements)$\times$ (number of generalized coordinates)

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Stiffness Matrix in Generalized Coordinates Base

The potential energy in the generalized base of coordinates (after substitution of $\mathbf{u} = \mathbf{A}_k\mathbf{q}$) yields the expression

$$E_p = \frac{1}{2}\sum_j k_j u_j^2 = \frac{1}{2}\mathbf{u}^T\{\mathbf{k}\}\mathbf{u} = \frac{1}{2}\mathbf{q}^T\mathbf{A}_k^T\{\mathbf{k}\}\mathbf{A}_k\mathbf{q} = \frac{1}{2}\mathbf{q}^T\mathbf{K}\mathbf{q}$$

As it can be seen, the potential energy is given by the positive-definite quadratic form of the Lagrange's generalized coordinates $\mathbf{q}$, where $\mathbf{K}$ is the stiffness matrix. Stiffness matrix is non-singular, symmetric and positive-definite, thus $\det\mathbf{K} > 0$ and can be achieved from formula

$$\mathbf{K} = \mathbf{A}_k^T\{\mathbf{k}\}\mathbf{A}_k$$

The flexibility matrix is inverse to stiffness matrix

$$\mathbf{D} = \mathbf{K}^{-1}$$

The kinetic energy

The kinetic energy of the whole system can be expressed by

$$E_k = \frac{1}{2}\sum_j m_j \dot{u}_j^2 = \frac{1}{2}\dot{\mathbf{u}}^T \{\mathbf{m}\}\dot{\mathbf{u}}$$

Inertia Matrix in Local Coordinates Base of Mass Centers

$$\{\mathbf{m}\} = diag(m_1 \quad m_2 \quad m_3 \quad \ldots) = \begin{bmatrix} m_1 & 0 & 0 & \cdots \\ 0 & m_2 & 0 & \cdots \\ 0 & 0 & m_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Transformation from Generalized to Local Coordinates

$$\mathbf{u} = \mathbf{A}_m \mathbf{q}$$

$\mathbf{A}_m$ is the transformation matrix whose elements depend on the configuration of the structure only; the dimension of this transformation matrix is (number of local coordinates of mass elements) × (number of generalized coordinates $n \geq d$)

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Inertia Matrix in Generalized Coordinates Base

The kinetic energy in the generalized base of coordinates (after substitution of $\mathbf{u} = \mathbf{A}_m \mathbf{q}$) yields the expression

$$E_k = \frac{1}{2}\sum_j m_j \dot{u}_j^2 = \frac{1}{2}\dot{\mathbf{u}}^T \{\mathbf{m}\}\dot{\mathbf{u}} = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{A}_m^T \{\mathbf{m}\}\mathbf{A}_m \dot{\mathbf{q}} = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{B}\dot{\mathbf{q}}$$

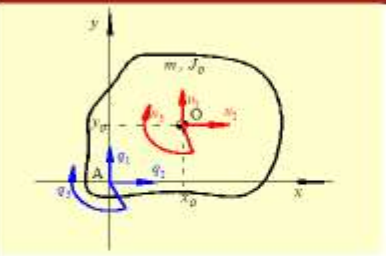
As it can be seen, the kinetic energy is given by the positive-definite quadratic form of the of Lagrange's generalized coordinates velocity, where

$$\mathbf{B} = \mathbf{A}_m^T \ \{\mathbf{m}\} \ \mathbf{A}_m$$

Where $\mathbf{B}$ is the square and symmetric matrix of inertia in a generalized coordinate base. In a minimal base of generalized coordinates $n = d$, matrix $\mathbf{B}$ is non-singular and positive-definite, thus $\det \mathbf{B} > 0$.

If $n > d$, inertia matrix is non-negatively definite and singular.

Illustrative Example



This is the plane rigid body, which mass is m and polar mass moment of inertia about the axe through the mass center point J_O is given. The example shows haw to determine a mass matrix in the generalized coordinate base.

$$\{\mathbf{m}\} = diag(m \quad m \quad J_O) = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J_O \end{bmatrix}$$

$$\mathbf{u} = \mathbf{A}_m \mathbf{q}$$

$$\mathbf{u} = [u_1 \quad u_2 \quad u_3]^T$$

$$\mathbf{q} = [q_1 \quad q_2 \quad q_3]^T$$

$$\mathbf{A}_m = \begin{bmatrix} 1 & 0 & -x_o \\ 0 & 1 & y_o \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B} = \mathbf{A}_m^T \{\mathbf{m}\} \mathbf{A}_m = \begin{bmatrix} m & 0 & -my_O \\ 0 & m & mx_O \\ -my_O & mx_O & m(x_O^2 + y_O^2) + J_O \end{bmatrix} = \begin{bmatrix} m & 0 & -S_y \\ 0 & m & S_x \\ -S_y & S_x & J_A \end{bmatrix}$$

$S_y = my_O$ static moment of mass about the axis x

$S_x = mx_O$ static moment of mass about the axis y

$J_A = m(x_O^2 + y_O^2) + J_O$ polar mass moment of inertia about the axis through the dynamic center point - A

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Damping Matrix in Generalized Coordinates Base

Rayleigh Dissipation Function describes the power of resistance forces, i.e. the work of these forces in time, which occurs in systems performing small oscillations. These forces are assumed to be proportional to velocities. The Rayleigh dissipation function, also known as the dissipation function, is given by the positive-definite quadratic form of the generalized velocities $\dot{\mathbf{q}}$

$$\Phi = \frac{1}{2}\sum_j c_j \dot{u}_j^2 = \frac{1}{2}\dot{\mathbf{u}}^T \{\mathbf{c}\} \dot{\mathbf{u}} \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{A}_d^T \{\mathbf{c}\} \mathbf{A}_d \dot{\mathbf{q}} = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{C} \dot{\mathbf{q}}$$

$$\{\mathbf{c}\} = diag(c_1 \quad c_2 \quad c_3 \quad \ldots) = \begin{bmatrix} c_1 & 0 & 0 & \cdots \\ 0 & c_2 & 0 & \cdots \\ 0 & 0 & c_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\mathbf{u} = \mathbf{A}_d \mathbf{q} \qquad \mathbf{C} = \mathbf{A}_d^T \{\mathbf{c}\} \mathbf{A}_d$$

$\mathbf{A}_d$ is the transformation matrix whose elements depend on the configuration of the structure only; the dimension of this transformation matrix is (number of local coordinates of damped elements)×(number of generalized coordinates $n \geq d$)

Excitation Forces

Work *W* is the transfer of energy that occurs when a force acts on a body, and is calculated as a dot product of the vector of force $\mathbf{F}$ and displacement $\mathbf{q}$ (displacement of the point the force is acting on). If a body is moving in such a way that the force has a component in a direction perpendicular to the direction of the body's motion, the work of that component is equal to zero. In any situation, the work is given by the linear form of the coordinates $\mathbf{q}$

$$W = \sum_j P_j u_j = \mathbf{u}^T \mathbf{P} = \mathbf{q}^T \mathbf{A}_t^T \mathbf{P} = \mathbf{q}^T \mathbf{F}$$

Excitation Force Vector in Local Coordinates Base $\mathbf{P} = [P_1 \quad P_2 \quad P_3 \quad \ldots]^T$

Excitation Force Vector in Generalized Coordinates Base $\mathbf{F} = \mathbf{A}_t^T \mathbf{P}$

$$\mathbf{u} = \mathbf{A}_t \, \mathbf{q}$$

$\mathbf{A}_t^T$ is the transformation vector from generalized to local coordinates

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Equation of Motion

After substitution of the into **Lagrangian Equations**, and after differentiation of these expressions with respect to each chosen coordinate, one can receive an number of equations of motion in the generalized coordinate base. In the matrix form, this system may be written as

$$\mathbf{B}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}(t)$$

Displacement Method

$$\mathbf{DB}\ddot{\mathbf{q}} + \mathbf{DC}\dot{\mathbf{q}} + \mathbf{q} = \mathbf{DF}(t)$$

Force Method