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Civil Engineering

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MATHEMATICS

A Short Introduction to Ordinary and Partial Differential Equations

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Preface

This lecture notes is a supporting material to the subject Mthematics in the framework of first semester graduate studies in *Civil Engineering* organised by Faculty of Civil Engineering of Wrocław University of Technology. General idea of the course is to give students an introduction to the most important problems in ordinary differential equations and some very basic ideas in partial differential equations with some important applications. I would like to emphasise that we are in the beginning of the implementing this course. Then this lectures note should be treated as a first step to help the students studying the applications of mathematics. It should generally cover the accepted syllabus, however it is expected that the lectures will modify the presented approach in the future. Therefore no one-to-one correspondence between this lecture note and the lectures should be expected.

Introduction
Remarks on notation

The first part of this lecture, containing Chapter 1, Chapter 2 and Chapter 3, is thought as a repetition of the most important ideas of the ordinary differential equations, which are necessary in understanding partial differential equations and boundary value problems. It is assumed, however, that the students are generally familiar with general concepts of ordinary differential equations, hence it can not be treated as a systematic course of the subject. The second part contains some selected elements of the partial differential equations theory with some applications oriented to civil engineering problems.

We begin this course with explaining some most important notations and definitions who will be useful within this notebook . The basic notations and definitions concerning partial differential equations will be commented in the Chapter 4.

Definition 0.1 (n-dimensional Euclidean space).

Let $\mathbf{R^n} = \{\mathbf{x} = (x_1, x_2, ...x_n) : x_i \in \mathbf{R}, i = 1,2,...,n\} = \mathbf{R} \times ... \times \mathbf{R}$ be the Cartesian product of n sets of real numbers. Consider a distance between points of $\mathbf{R^n}$ defined by the equation

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} \tag{01}$$

Consider now ($\mathbf{R^n}$**, +, · ,** d), where + is the addition of vectors and · multiplication by scalars (real numbers). The ($\mathbf{R^n}$**, +, · ,** d) is called the *n-dimensional Euclidean space*.

Remark 01.

The distance given by eqn. (01) can be considered as generated by scalar product of the form:

$$\mathbf{x} \circ \mathbf{y} = \sum_{i=1}^{n} x_i y_i \tag{02}$$

Compare general theory of unitary spaces ([1]).

Definition 02.

A sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ of elements of $\mathbf{R^n}$ is said to be convergent to $\mathbf{x}_o \in \mathbf{R^n}$ if and only if

$$\lim_{n \to \infty} d(\mathbf{x}_n, \mathbf{x}_o) = 0 \tag{03}$$

The convergence defined by the equation (03) is equivalent to the following one:

$$\lim_{n \to \infty} x_{ni} = x_{oi} \quad , \forall i = 1,2,...,n \tag{04}$$

Definition 03.

A neighborhood of the point $\mathbf{x}_o$ of the radius r is the following set

$$B(\mathbf{x}_o, r) = \{\mathbf{x} \in \mathbf{R^n} : d(\mathbf{x}, \mathbf{x}_o) < r\} \tag{05}$$

An alternative name of this set is an open ball with the center $\mathbf{x}_o$ and radius r.

Definition 04.

A set $U \subset \mathbf{R^n}$ is said to be an open set if and only if

$$\forall\, \mathbf{x}_o \in U \quad \exists\, r > 0 \quad B(\mathbf{x}_o, r) \subset U \tag{06}$$

Definition 05.

A set $B \subset \mathbf{R}^\mathbf{n}$ is a closed set if and only if for any sequence $\{x_k\} \subset B$ the following implication holds:

$$\lim_{k \to \infty} x_k = x_0 \quad \Rightarrow \quad x_0 \in B \tag{07}$$

It can be proved that a set B is closed if and only if $\left(\mathbf{R}^\mathbf{n} - B\right)$ is open.

Definition 06.

The closure of a set B consists of all points in B plus the limit points (in the sense of (07)) of B. The closure is denoted by $\overline{B}$.

Intuitively, these are all the points that are "near" B.

Definition 07.

The interior of the set B, denoted as Int(B) is the largest (in the sense of inclusion) open subset of B.

Definition 08.

The boundary, ∂B , of the set B is the following set

$$\partial B = B - \mathrm{Int}(B) \quad . \tag{08}$$

In order to make readers familiar with different types of notations that can be found in the literature two ways of notations will be presented within the further course. Namely the derivatives are denoted by

$\frac{dy(x)}{dx}, \frac{d^2 y(x)}{dx^2}, ..., \frac{d^{n-1} y(x)}{dx^{n-1}}, \frac{d^n y(x)}{dx^n}$. These are first second , (n-1)th and nth order derivatives, respectively. All of the are derivatives with respect x variable, which is independent variable in this case. This kind of notation is the classical Leibnitz’s notation. It is suits very well the separation variable problems.

Alternatively the same derivatives will be denoted by $y', y'', ..., y^{(n-1)}, y^{(n)}$.

In the first chapter that concerns ordinary differential equations the independent variables will be denoted by x, t. In first three chapters only ordinary differential equations are under consideration, then the definitions 01 to 05 are reduced to one-dimensional case of set of the real numbers **R**.

Chapter 1
Basic ideas of ordinary differential equations

Definition 1.1.
Ordinary differential equation is an equation of the form:

$$G\left(x, y, y', y'',..., y^{(n)}\right) = 0 \quad , \tag{1.1}$$

where
$G : R^{n+1} \to R$ is a given function, $y(x)$ is the unknown function of the variable x and at least one of the derivatives $y', y'',...y^{(n)}$ appears in it.
If the derivative $y^{(n)}$ appears in the equation and simultaneously there is no derivative of any higher order in the equation, then the equation is called an equation of the n-th order.

Example 1.1.
Consider the following equation:

$$\frac{dy}{dx} = f(x) \quad , \tag{1.2}$$

Where f: R R is a continuous on interval [a,b]. Let F denote an antiderivative of f on the interval [a,b] .
Then the solution of eqn (2) is given by:

$$y(x) = F(x) + C \quad , \tag{1.3}$$

where $C \in \mathbf{R}$ is a constant. Assuming that for $x_0 \in (a, b)$ solution (3) satisfies the condition

$$y(x_0) = y_0 \tag{1.4}$$

Combining (1.3) and (1.4) one gets

$$C = y_0 - F(x_0) \tag{1.5}$$

Then the solution of (1.2), which satisfies the condition (1.4) takes the form

$$y(x) = F(x) + y_0 - F(x_0) \tag{1.6}$$

or

$$y(x) = y_0 + \int_{x_0}^{x} f(t)dt \tag{1.7}$$

If, for example, equation (1.2) takes the form

$$\frac{dy}{dx} = 3x^2 \quad , \tag{1.8}$$

in conjunction with initial condition $y(0)=1$. Then, by (1.6), the solution takes the form

$$y(x) = x^3 + 1 \ . \tag{1.9}$$

In order to avoid confusions it is necessary to define some most important objects.

Definition 1.2.
Given a differential equation of the form (1.1). A function y: $I \subset \mathbf{R} \to \mathbf{R}$ is called the **solution** or **integral curve** for G, if y is n-times differentiable on I, and

$$G\left(x, y, y', y'',..., y^{(n)}\right) = 0 \quad \text{for each} \quad x \in I \ . \tag{1.10}$$

Given two solutions u: $J \subset \mathbf{R} \to \mathbf{R}$ and y: $I \subset \mathbf{R} \to \mathbf{R}$, u is called an **extension** of v if $I \subset J$ and

$$u(x) = y(x) \quad x \in I \ . \tag{1.11}$$

A solution which has no extension is called a **global solution**. A **general solution** of an n-th order equation is a solution containing n arbitrary variables, corresponding to n constants of integration. A **particular solution** is derived from the general solution by setting the constants to particular values, often chosen to fulfill set of initial conditions or boundary conditions. A singular solution is a solution that can't be derived from the general solution.
Example 1.2.
Find the curve, which includes the point A(0,-2), such that the slope of the tangent at each point is equal to the triple value of ordinate of the point A.
As the slope of the tangent to curve at a given point is the value of the first derivative at this point, we can write

$$\frac{dy}{dx} = 3y \ , \tag{1.12}$$

or using another notation

$$\frac{1}{y} y' = 3 \ . \tag{1.13}$$

By integrating both sides of (1.13) one gets

$$\int \frac{1}{y} dy = \int 3dx \quad \Rightarrow \ln|y| = 3x + C_1, \qquad y = C_2 e^{3x} \ . \tag{1.14}$$

The last expression in eqn. (1.14) is the general solution of the eqn. (1.12). To solve the problem stated in the beginning it is sufficient to employ the assumption that the integral curve should cross the point A(0,-2). Hence $y(0) = -2$ and consequently

$$-2 = C_2 e^0 \Rightarrow C_2 = -2, \quad \textit{and} \quad y = -2e^{-3x} \quad \textit{for} \quad x \in \quad (-\infty, +\infty) \ . \tag{1.15}$$

Definition 1.3.
Consider the following problem:
Find a particular solution of an *n*-th order differential equation, (1.1), which satisfies given set of initial conditions

$$y(x_0) = y_0, \quad y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}. \tag{1.12}$$

The above problem is called the Cauchy problem.
Example 1.4.
Find the solution of the following Cauchy problem

$$y'' = 6x, \quad and \quad y(0) = 0, \quad y'(0) = 1 \quad . \tag{1.13}$$

By successive integration of the differential equation one finds

$$y'' = 6x \quad \Rightarrow \quad y' = 3x^2 + C_1 \Rightarrow \quad y = x^3 + C_1 x + C_2 \tag{1.14}$$

By substituting initial conditions one gets

$$0 + C_1 \cdot 0 + C_2 = 0, \quad and \quad 3 \cdot 0 + C_1 = 1 \quad \Rightarrow \quad C_1 = 1; \quad C_2 = 0 \tag{1.15}$$

Inserting constants to the last equation in (1.14) gives the solution to the problem as

$$y = x^3 + x \quad . \tag{1.16}$$

The next example demonstrates checking the general solution of an ordinary differential equation.
Example 1.5.
Let us demonstrate that the family of functions of the form

$$y = C_1 \sin x + C_2 \cos x \,, \tag{1.17}$$

where C_1 and C_2 are real constants, constitutes the a general solution of the equation

$$y''(x) + y(x) = 0, \quad x \in R \quad . \tag{1.18}$$

Assume that $x_{0,} y_{0,} y_1 \in R$. Let us define initial conditions as

$$y(x_0) = y_0 \,; \quad y'(x_0) = y_1 \quad . \tag{1.19}$$

First let us check that a function of the form (1.17) satisfies the equation (1.18). The first and second derivatives of function (1.17) are as follows

$$y' = C_1 \cos x - C_2 \sin x, \quad y'' = -C_1 \sin x - C_2 \cos x \quad . \tag{1.20}$$

Substitution eqn. (1.17) and the second eqn. in (1.20) into left hand side of the eqn. (1.18) gives

$$y''(x) + y(x) = -C_1 \sin x - C_2 \cos x + C_1 \sin x + C_2 \cos x = 0 . \tag{1.21}$$

Then (1.18) is satisfied. Let us now substitute initial conditions (1.19) into (1.17) and the first eqn. of (1.20)

$$C_1 \sin x_0 + C_2 \cos x_0 = y_0; \qquad C_1 \cos x_0 - C_2 \sin x_0 = y_1 \ . \tag{1.22}$$

Solving eqs. (1.22) with respect C_1 and C_2 gives

$$C_1 = y_0 \sin x_0 + y_1 \cos x_0; \qquad C_2 = y_0 \cos x_0 - y_1 \sin x_0 \ . \tag{1.23}$$

Finally inserting the solution (1.23) into (1.17) gives

$$y = (y_0 \sin x_0 + y_1 \cos x_0) \sin x + (y_0 \cos x_0 - y_1 \sin x_0) \cos x \quad . \tag{1.24}$$

It means that constants are determined In a unique way. This shows that the conditions given in the definition 1.2 are fulfilled and the family of functions given by (1.17) is the general solution of the differential equation (1.18).

When solving differential equations an important problem is the existence and uniqueness of solutions. The question of existence and uniqueness can be a vital one in many numerical solutions carried out in mechanics. In numerical computations certain specific algorithms (usually iterative procedures) are created in order to approximate solution. Therefore the existence of solution should be guaranteed. Moreover it essential to know to which solution the iterative procedure is convergent. Therefore if the solution is unique the problem under consideration becomes simpler. Below an example of existence and uniqueness theorem is given. This theorem is one of the most important for the ordinary differential equations of the first order.

Theorem 1.1.

Consider the first order ordinary differential equation

$$y' = f(x, y) \quad , \tag{1.25}$$

where f is a continuous function on a rectangle $D = [a,b] \times [c,d]$ in the, x,y – plane (**R**[2]).

Assume that its partial derivative $\frac{\partial f}{dy}$ is defined and continuous on Int(D). Then for any point $(x_0, y_0) \in \mathrm{Int}(D)$, there exists exactly one solution of the Cauchy problem

$$y' = f(x, y) \quad and \quad y(x_0) = y_0 \qquad .$$

The proof of this theorem can be found in many monographs of the subject, e.g. [2].

Chapter 2
Most important classes of the ordinary differential equations of the first order

2.1. Equations of separated variables

Let f be a continuous function on an interval (a, b) and *h* be a continuous function on an interval (c, d). Moreover, assume that *h* does not take zero value on (c, d).

Definition 2.1.

Let $x \in (a,b)$ and $y(x) \in (c,d)$. Equation

$$\frac{dy}{dx} = \frac{f(x)}{h(y)}, \tag{2.1}$$

Which can be equivalently written as

$$h(y)dy = f(x)dx \quad , \tag{2.2}$$

where the unknown function is *y*(*x*), is called an **equation of separated variables**.

Treating the derivatives in the eqn. (2.1) as a quotient of two differentials one can carry out the following transformations:

$$\{h[y(x)]y'(x) - f(x)\}dx = 0 \quad \Rightarrow \quad \int h(y)dy = \int f(x)dx + C \quad \Rightarrow \quad , \tag{2.3}$$
$$\Rightarrow \quad d\{H[y(x)] - F(x)\} = 0 \quad \Rightarrow \quad H[y(x)] - F(x) = C$$

where

$$H(y) = \int h(y)dy; \quad F(x) = \int f(x)dy \tag{2.4}$$

and *C* is a real constant.

The explanation given above shows that the following theorem holds.

Theorem 2.1.

Let f be a continuous function on an interval (a, b) and *h* be a continuous function on an interval (c, d). Moreover, assume that *h* does not take zero value on (c, d). Then the equation given below

$$\int h(y)dy = \int f(x)dx + C \quad , \tag{2.5}$$

where *C* is a real constant, gives the general solution of the equation (2.1). Moreover, each point (*x*, *y*) of the rectangular $P = \{(x,y) : x \in (a,b) \wedge y \in (c,d)\}$ belongs to only one integral curve of the equation (2.1).

This way the general solution of the equation of separated variables is known provided that integrals in (2.5) are easy for evaluating. An example of solving this kind of equation is given below.

Example 2.1.

Find the integral curve of the equation

$$\frac{dy}{dx} = -\frac{2x}{y} \quad , \tag{2.6}$$

crossing the point *P*(1,1).

By separating variables and integrating one gets

$$y dy = -2x dx \quad \Rightarrow \quad \frac{y^2}{2} = -x^2 + C \quad \Rightarrow \quad \frac{y^2}{2} + x^2 = C \tag{2.7}$$

The last equation of (2.7) shows that the integral curves of the eqn. (2.6) are ellipses. Inserting to this equation coordinates of the point *P* leads to evaluating the constant $C = 3/2$. Hence the ellipse crossing point *P* is given by the equation

$$\frac{y^2}{3} + \frac{2}{3}x^2 = 1 \tag{2.8}$$

Please note that replacing the constant 2 in eqn. (2.6) by 1 leads to integral curve in the form of circle. Generalising the eqn. (2.6) to the form

$$\frac{dy}{dx} = -k\frac{x}{y} \tag{2.9}$$

one gets different shape of integral curves Goering by a value of k. Namely, the integral curve can be a circle, an ellipse or a hyperbola if k = 1, k > 0, k < 0, respectively.

2.2. Homogeneous equation

Let $f(u)$ be a continuous function on (a,b), which satisfies the condition $f(u) \neq 0$ in (a,b). A first order differential equation of the form

$$\frac{dy}{dx} = f(u) \tag{2.10}$$

Is called the homogeneous equation. Any homogeneous equation can be reduced to the an equation of separated variables by applying the described below procedure. Consider the substitution

$$u(x) = \frac{y(x)}{x} \quad \Rightarrow \quad y = xu\,. \tag{2.11}$$

Consequently the derivative *y*' can be written as

$$\frac{dy}{dx} = \frac{d}{dx}(xu) = u + x\frac{du}{dx}\,. \tag{2.12}$$

Substituting eqs (2.11) and (2.12) into (2.10) one gets the equation

$$u + x\frac{du}{dx} = f(u)\,, \tag{2.13}$$

which can be transformed to the form

$$\frac{du}{f(u) - u} = \frac{dx}{x} \qquad . \tag{2.14}$$

The above equation is a equation of separable variables. Hence

$$\int \frac{du}{f(u) - u} = \int \frac{dx}{x} + C = \ln|x| + C\,, \tag{2.15}$$

where C is a real constant. The further evaluation is possible if an explicit form of the function f is known. This is demonstrated by the example given below.

Example 2.2.

Find integral curves corresponding to the equation

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}. \tag{2.16}$$

Egn. (2.16) can be written as

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{y}{x} - \frac{x}{y}\right). \tag{2.17}$$

Now it is clear that the equation is a homogeneous equation. According to general procedure described above substitution $y = xu$ transforms the equation under consideration to the form

$$u + x\frac{du}{dx} = \frac{1}{2}\left(u - \frac{1}{u}\right). \tag{2.18}$$

The above equation is equivalent to

$$x\frac{du}{dx} = -\frac{1}{2}\frac{u^2 + 1}{u} \tag{2.19}$$

And hence

$$-\frac{2u}{u^2 + 1}du = \frac{dx}{x} \qquad . \tag{2.20}$$

Integrating left hand side with respect to variable u and right hand side with respect variable x leads to

$$\ln|x| = -\int \frac{2u}{u^2 + 1}du + C\,. \tag{2.21}$$

The right hand side can be reduced as follows

$$-\int \frac{2u}{u^2 + 1}du + C = -\ln\left(u^2 + 1\right) + C = \ln\frac{C_1}{u^2 + 1}. \tag{2.22}$$

Hence

$$\ln|x| = \ln\frac{C_1}{u^2 + 1} \quad \Rightarrow \quad |x| = \frac{C_1}{u^2 + 1}. \tag{2.23}$$

Because $y = xu$ therefore, due to (2.23),

$$|xu| = \frac{C_1|u|}{u^2+1} \quad \Rightarrow \quad |y| = \frac{C_1|u|}{u^2+1} \quad \Rightarrow \quad x^2+y^2 = \frac{C_1^2 + C_1^2u^2}{\left(u^2+1\right)^2} = C_1\frac{1}{u^2+1} = C_1x \quad \Rightarrow$$
$$\Rightarrow \left(x - \frac{C_1}{2}\right)^2 + y^2 = \frac{C_1^2}{4} \quad . \qquad (2.24)$$

The last equation in (2.24) is an equation of circle. This means that the integral curves of eqn. (2.16) are circles with centers at $\left(\frac{C_1}{2}, 0\right)$ and radii equal to $\frac{C_1}{2}$. These circles are tangent to the *y*- axis at the origin.

2.3. Linear equation of the first order

Among ordinary differential equations linear equations play the central role. Due to their simple form the general theory and methods of solving are well-developed. On the other hand most simpler problems of classical mechanics are governed by linear differential equations.

Definition 2.2.

An equation of the form:

$$y' + p(x)y = q(x) \quad , \qquad (2.25)$$

where $p(x)$ and $q(x)$ are continuous functions defined on an interval $[a,b] \subset \mathbf{R}$, is called the **linear differential equation (lde)** of the first order. If the right hand side of the equation (2.25) equals zero, the equation is called a **homogeneous linear differential equation (hlde)**. Consider hlde

$$y' + p(x)y = 0 \,. \qquad (2.26)$$

Note that the function

$$y_0(x) \equiv 0 \qquad (2.27)$$

is a solution of (2.26), which is called the trivial solution. In the sequel we will seek for a nontrivial solution, i.e. for a function satisfying (2.26) which is not identically equal to zero. It is easy to see that the equation (2.27) is an equation of separated variables

$$\frac{dy}{y} = -p(x)dx \qquad (2.28)$$

Integrating the left hand side of eqn. (2.28) with respect to variable *y* and the Wright hand side with respect to *x* one gets

$$\ln|y| = -\int p(x)dx + \ln C \quad , \qquad (2.29)$$

and hence the general solution of the eqn. (2.26) is of the form

$$y = c_1 e^{-\int p(x)dx} \quad . \qquad (2.30)$$

Computations carried out above allows formulating the theorem concerning solution of hlde.

Theorem 2.2.
Let us consider a linear homogeneous differential equation of the form (2.26). Assume that $p(x)$ is a continuous function defined on an interval (a, b). Then eqn. (2.30) determines the general solution of hlde (2.26). Moreover, each point (x, y) of the open set $D = \{(x,y): x \in (a,b) \wedge y \in (-\infty, +\infty)\}$ belongs to only one integral curve of the equation (2.26).

Having given the general solution of hlde one can construct the general solution of non-homogeneous lde (2.25) by applying so-called the **method of variation of constants**. This can be done in following way. Assume that the general solution can be found in the form

$$y(x) = C(x) \cdot e^{-\int p(x)dx} \quad . \tag{2.31}$$

The function given by (2.31) looks similar to that given by (2.30). In (2.31), however, C is not a constant value, but another function of the independent variable x. The aim is to determine function $C(x)$ such that the equation (2.31) will give a general solution of lde under consideration. The derivative of the function (2.31) is given by

$$\frac{dy}{dx} = C'(x) \cdot e^{-\int p(x)dx} + C(x)e^{-\int p(x)dx}[-p(x)] \quad . \tag{2.32}$$

Substitution (2.31) and (2.32) to eqn. (2.25) Leeds to

$$C'(x)e^{-\int p(x)dx} + C(x)e^{-\int p(x)dx}[-p(x)] + p(x) \cdot C(x)e^{-\int p(x)dx} = q(x) \tag{2.33}$$

Solving eqn. (2.33) with respect to $C'(x)$ gives

$$C'(x) = q(x)e^{\int p(x)dx} \quad . \tag{2.34}$$

Integrating (2.34) with respect to the variable x the function $C(x)$ can be obtained

(2.31) leads to general solution of non-homogeneous lde

$$y(x) = C_1 e^{-\int p(x)dx} + e^{-\int p(x)dx} \int q(x)e^{\int p(x)dx} dx \tag{2.35}$$

The above considerations leads to the theorem concerning the solution of lde
Theorem 2.3.
Let us consider a linear differential equation of the form (2.25). Assume that $p(x)$ and $q(x)$ are continuous function defined on an interval (a, b). Then eqn. (2.35) determines the general solution of lde (2.25). Moreover, each point (x, y) of the open set $D = \{(x,y): x \in (a,b) \wedge y \in (-\infty, +\infty)\}$ belongs to only one integral curve of the equation (2.25).
The first statement in this theorem has been proved above. The proof of the second statement can be found in many monographs.
Consider now an example of application of the method of variation of constants.
Example 2.3.
Find the general solution of the non-homogeneous linear equation

$$y'(1-x^2)+xy=2x \tag{2.36}$$

Dividing by $(1-x^2)$ one gets

$$y'+\frac{xy}{1-x^2}=\frac{2x}{1-x^2} \tag{2.37}$$

Consider first the corresponding homogeneous equation

$$y'+\frac{xy}{1-x^2}=0 \tag{2.38}$$

By virtue of the Theorem 2.2 the general solution of the eqn. (2.38) is given by

$$y=c_1e^{-\int\frac{x}{1-x^2}dx}=c_1e^{\frac{1}{2}\ln\left|1-x^2\right|}=c\sqrt{\left|1-x^2\right|} \tag{2.39}$$

Let us apply the method of variation of constant assuming that we are looking for the solution of the (2.37) in the form

$$y(x)=C(x)\cdot\sqrt{1-x^2}\quad . \tag{2.40}$$

First let us find the derivative of the function given by eqn. (2.40).

$$y'(x)=C'\cdot\sqrt{1-x^2}+C(x)\cdot\frac{\frac{1}{2}\cdot(-2x)}{\sqrt{1-x^2}}=C'(x)\sqrt{1-x^2}-\frac{C(x)\cdot x}{\sqrt{1-x^2}} \tag{2.41}$$

Next step is the substitution of (2.40) and (2.41) into (2.37)

$$C'(x)\cdot\sqrt{1-x^2}-\frac{C(x)\cdot x}{\sqrt{1-x^2}}+\frac{x\cdot C(x)\sqrt{1-x^2}}{1-x^2}=\frac{2x}{1-x^2} \tag{2.42}$$

Next transformations lead to the general solution of

$$\begin{gathered}C'(x)=\frac{2x}{(1-x^2)^{\frac{3}{2}}}\Rightarrow C(x)=\int\frac{2xdx}{(1-x^2)^{\frac{3}{2}}}\Rightarrow C(x)=\frac{2}{\sqrt{1-x^2}}+C_1\\ y(x)=(\frac{2}{\sqrt{1-x^2}}+C_1)\cdot\sqrt{1-x^2}=2+C_1\sqrt{1-x^2}\end{gathered} \tag{2.43}$$

After some simple rearrangements the integral curves in the eqn. (2.43) take the form

$$y = 2 + C_1\sqrt{1-x^2} \quad \Rightarrow \quad y - 2 = C_1\sqrt{1-x^2} \quad \Rightarrow \quad \frac{(y-2)^2}{C_1^2} = 1 - x^2 \quad \Rightarrow$$

$$\Rightarrow \quad (y-2)^2 = C_1^2(1-x^2) \quad \Rightarrow \quad \frac{(y+2)^2}{C_1^2} + x^2 = 1 \qquad (2.44)$$

It easy to observe that the final equation in (2.44) is a family of ellipses.
The next example concerns of finding of family of integral curves of non-homogeneous linear ordinary equation. However, the way of solving the problem is not a conventional one.
Example 2.4.
Find the family of integral curves of the equation

$$\frac{dy}{dx} = \frac{1}{x\cos y + \sin 2y} \ . \qquad (2.45)$$

The above equation is neither linear nor of separated variables. In this case instead looking for integral curves in the form $y(x)$ we will find integral curves in the form of $x(y)$. Please note that in the (x,y)-plane $y(x)$ and $x(y)$ are the same curves. Therefore, by changing numerator with denominator (upside down procedure) in both sides of the eqn. (2.45), we obtain

$$\frac{dx}{dy} = x\cos y + \sin 2y \, . \qquad (2.46)$$

The equation (2.46) is a non-homogeneous linear equation with respect the unknown function $x(y)$. Hence the routine procedures described above can be applied. The solution of the corresponding homogeneous equation

$$\frac{dy}{dx} = x\cos y \qquad (2.47)$$

is obtained by means of the formula (2.30) in the form

$$x = Ce^{\sin y} \quad . \qquad (2.48)$$

Then applying variation of constant procedure one gets

$$x = C(y)e^{\sin y} \quad \Rightarrow \quad \frac{dx}{dy} = C'(y)e^{\sin y} + C(y)e^{\sin y}\cos y \quad \Rightarrow$$

$$\Rightarrow \quad C'(y)e^{\sin y} + C(y)e^{\sin y}\cdot\cos y - C(y)e^{\sin y}\cos y = \sin 2y \quad \Rightarrow \qquad (2.49)$$

$$\Rightarrow \quad C'(y)e^{\sin y} = \sin 2y \;\Rightarrow\; C'(y) = e^{\sin y}2\sin y \;\Rightarrow\; C(y) = \int e^{-\sin y}\sin 2y dy + C_1$$

To evaluate the last integral in (2.49) the substitution $z = \sin y \Rightarrow dz = (\cos y)dy$ and next integration by parts can be applied, namely

$$\int e^{-\sin y} 2\sin y\cos y dy = \int e^{-z}\cdot 2z dz = 2\left(-e^{-z}\cdot z + \int e^{-z}dz\right) = -2e^{-z}\cdot z - 2e^{-z} + C =$$
$$= -2e^{-\sin y}\cdot\sin y + \left(-2e^{-\sin y}\right) + C = -2e^{-\sin y}(1+\sin y) + C \ . \qquad (2.50)$$

Then the final formulae for $C(y)$ is

$$C(y) = -2e^{-\sin y}(1 + \sin y) + C_2 . \qquad (2.51)$$

Now combining (2.48) and (2.51) the general solution of (2.46) can be written as

$$x(y) = C_2 e^{\sin y} - 2\sin y - 2 \quad . \qquad (2.52)$$

Equation (2.52) describes the family of integral curves of the eqn. (2.45).

The method of constant variation gives an universal tool for solving ordinary linear differential equations. However, there some other methods that can be useful in practical computations. One of the is the **method of undetermined coefficients,** which is sometimes called **the lucky guess method**. The method bases on the following theorem:

Theorem 2.4.

Consider lde of the form (2.25). Assume that y_h is the general solution of the corresponding homogeneous equation (2.26) and y_p is any particular solution of the eqn (2.25). Then the general solution y to the equation (2.26) would be

$$y = y_h + y_p \quad . \qquad (2.53)$$

The use of this methods will be explained by examples given below.

Example 2.5.

Find the general solution to the equation

$$\frac{dy}{dx} + 4y = x^3 \quad . \qquad (2.54)$$

According to Theorem 2.2 the general solution to hlde corresponding to (2.54) is the following

$$y = Ce^{-4x} \qquad (2.55)$$

Because the equation (2.54) has a constant coefficients on left hand side and its right hand side is a polynomial of the third degree, we can predict a particular solution of (2.54) in the form of the polynomial of the third degree, namely

$$y_1 = Ax^3 + Bx^2 + Cx + D \quad . \qquad (2.55)$$

Evaluating derivative of y_1 as $y_1' = 3Ax^2 + 2Bx + C$ and substituting y_1 and y_1' into (2.54) one gets

$$4Ax^3 + x^2(3A + 4B) + (2B + 4C)x + (C + 4D) = x^3 \qquad (2.56)$$

The polynomials of both sides are identical if the corresponding coefficients are mutually equal. Hence

$$\begin{cases} 4A = 1 \quad \Rightarrow \quad A = \dfrac{1}{4} \\ 3A + 4B = 0 \quad \dfrac{3}{4} + 4B = 0 \Rightarrow 4B = -\dfrac{3}{4} \Rightarrow B = -\dfrac{3}{16} \\ 2B + 4C = 0 \quad -\dfrac{6}{16} + 4C = 0 \quad \Rightarrow \quad C = \dfrac{3}{32} \\ C + 4D = 0 \qquad D = -\dfrac{1}{4}C = -\dfrac{3}{128} \end{cases} \tag{2.57}$$

Inserting computed above constant into (2.55) and then combining (2.53), (2.55) and (2.56) one gets the general solution of the (2.54)

$$y_1(x) = Ce^{-4x} + \frac{1}{4}x^3 - \frac{3}{16}x^2 + \frac{3}{32}x - \frac{3}{128}\,. \tag{2.58}$$

The lucky guess method give satisfactory results not only in the case when the right hand side of the equation has a form of polynomial. The example below shows another case of right hand side of a linear equation.

Example 2.6.

Find a particular solution of the equation

$$\frac{dy}{dx} + 2y = xe^x\,, \tag{2.59}$$

Which satisfies the initial condition $y(0) = 2$.

Due to eqn. (2.30) it is easy to see that the general solution of the corresponding homogeneous solution is

$$y = Ce^{-2x} \tag{2.60}$$

As the right hand side of (2.59) is a product of a polynomial and an exponential function we will seek a particular solution in the form

$$y_1 = (Ax + B)e^x \tag{2.61}$$

The way of solving is analogical to the previous one. First the derivative of y_1 is evaluated

$$y_1' = Ae^x + (Ax + B)e^x \tag{2.62}$$

Then y_1 and y_1' are inserted into (2.59) and after some rearrangement in terms one gets

$$e^x \cdot x(A + 2A) + e^x(A + B + 2B) = xe^x \tag{2.63}$$

The comparison of corresponding coefficients leads to

$$
\begin{aligned}
&3A = 1; \qquad A + 3B = 0 \\
&A = \frac{1}{3} \Rightarrow B = -\frac{1}{3}A = -\frac{1}{9}
\end{aligned}
\qquad . \tag{2.64}
$$

Inserting constants *A* and *B* into (2.61) determines the particular solution of (2.59) as

$$
y_1 = \frac{1}{3}\left(x - \frac{1}{3}\right)e^x \qquad . \tag{2.65}
$$

Due to Theorem 2.4 the sum of functions given by (2.60) and (2.65) is the general solution of (2.59)

$$
y = Ce^{-2x} + \frac{1}{3}\left(x - \frac{1}{3}\right)e^x \qquad . \tag{2.66}
$$

Inserting the initial condition $y(0) = 2$ into (2.66) allows to determine the constant *C*. Finally the particular solution under consideration is

$$
y = \frac{19}{9}e^{-2x} + \frac{1}{3}\left(x - \frac{1}{3}\right)e^x \qquad . \tag{2.67}
$$

The method of undetermined coefficients can find more applications in conjunction with so-called **superposition principle.** The superposition principle play vital role in mechanics when linear problems are under consideration and it finds many applications in structural mechanics. We formulate this principle as the theorem below.

Theorem 2.5. (Superposition principle)

Assume that y_{p1} is a particular solution of the equation

$$
\frac{dy}{dx} + p(x)y = f_1(x) \qquad . \tag{2.68}
$$

Assume then y_{p2} is a particular solution of the equation

$$
\frac{dy}{dx} + p(x)y = f_2(x) \qquad . \tag{2.69}
$$

Then $y_p = y_{p1} + y_{p2}$ is a particular solution of the equation

$$
\frac{dy}{dx} + p(x)y = f_1(x) + f_2(x) \qquad . \tag{2.70}
$$

The next example will demonstrate the usefulness of the above theorem.

Example 2.7.

Find the particular solution of the equation

$$
\frac{dy}{dx} + 3y = x^2 - \cos 3x \qquad , \tag{2.71}
$$

which satisfies the initial condition $y(0) = \frac{49}{54}$.

Applying (2.30) the general solution of the corresponding homogeneous equation is fund as

$$y = -Ce^{-3x} \quad . \tag{2.72}$$

Consider now two non-homogeneous linear differential equations

$$\frac{dy}{dx} + 3y = x^2 \quad ; \quad \frac{dy}{dx} + 3y = -\cos 3x \tag{2.73}$$

For the first equation we will seek a particular solution in the form of a polynomial of the second degree, i.e.

$$y_1 = Ax^2 + Bx + C \tag{2.74}$$

Analogically to the Example 2.4 computing the first derivative of (2.74), substituting it to the first equation of (2.73) together with the function y_1 leads to the following equation

$$3Ax^2 + (2A + 3B)x + B + 3C = x^2 \quad . \tag{2.75}$$

Comparing corresponding coefficients gives the following system of algebraic equations

$$3A = 1; \quad 2A + 3B = 0; \quad B + 3C = 0 \quad . \tag{2.76}$$

Solving the system (2.76) and substituting constant A, B and C into eqn. (2.74) gives final form of the particular solution y_1

$$y_1 = \frac{1}{3}x^3 - \frac{2}{9}x + \frac{2}{27} \quad . \tag{2.77}$$

Now we hale to find a particular solution of the second equation in (2.73). As its right hand side is of the form of linear combination of trigonometric functions we will seek for a particular solution of the form

$$y_2 = A\sin 3x + B\cos 3x \, . \tag{2.78}$$

The algorithm is the same. Find the first derivative y_2' of y_2 and substitute y_2' and y_2 into the second equation of (2.73). These steps give the following equation

$$3A\cos 3x - 3B\sin 3x + 3(A\sin 3x + B\cos 3x) = -\cos 3x \quad . \tag{2.79}$$

And after rearrangements

$$\cos 3x(3A + 3B) + (3A - 3B)\sin 3x = -\cos 3x \, . \tag{2.80}$$

Comparison of coefficients corresponding to sine and cosine leads to the system of algebraic equations

$$3A + 3B = -1 \quad , \quad 3A - 3B = 0 \qquad . \tag{2.81}$$

Solving the above system with respect A and B and inserting the solution into (2.78), one obtain the following particular solution y_2

$$y_2 = -\frac{1}{6}\sin 3x - \frac{1}{6}\cos 3x \qquad . \tag{2.82}$$

Applying now the superposition principle, we obtain the general solution of the equation (2.71) as the sum of solutions given by eqs. (2.73), (2.77) and (2.82). Hence the final result is

$$y = Ce^{-3x} + \frac{1}{3}x^3 - \frac{2}{9}x - \frac{2}{27} - \frac{1}{6}\sin 3x - \frac{1}{6}\cos 3x \qquad . \tag{2.83}$$

By utilising the initial condition the constant C is determined and the particular solution which fulfils that condition is

$$y = e^{-3x} + \frac{1}{3}x^3 - \frac{2}{9}x + \frac{2}{27} - \frac{1}{6}\sin 3x - \frac{1}{6}\cos 3x \, . \tag{2.84}$$

Presented above methods of solving non-homogeneous linear differential equations base on two steps procedures. In both cases (the method of constant variation and method of undetermined coefficients) the as the first step the general solution of the corresponding homogeneous differential equation has to be found. Now we demonstrate a method which straightforward leads to solution of non-homogeneous equation omitting solving the homogeneous one. This method is called **the method of integrating factor** and in many case can be the most efficient. Consider now a linear equation of the form (2.25). Let us multiply both sides of (2.25) by the following nonnegative expression

$$\exp\left(\int p(x)dx\right) \rangle 0 \qquad . \tag{2.85}$$

The left hand side of the above expression, which is any fixed antiderivative of the function $p(t)$, is called an integrating factor. The both sides multiplication leads to

$$y'(x)\exp\left(\int p(x)dx\right) + p(x)y(x)\exp\left(\int p(x)dx\right) = q(x)\exp\left(\int p(x)dx\right) \qquad . \tag{2.86}$$

It is easy to observe that the left hand side of the eqn. (2.86) is the derivative of the product of two functions as

$$\frac{d}{dx}\left[y(x)\exp\left(\int p(x)dx\right)\right] \qquad . \tag{2.87}$$

Combining (2.86) and (2.87) one gets

$$\frac{d}{dx}\left[y(x)\exp\left(\int p(x)dx\right)\right] = q(x)\exp\left(\int p(x)dx\right) \quad . \tag{2.88}$$

Integrating both sides of the (2.88) one obtain

$$y(x)\exp\left(\int p(x)dx\right)=\int q(x)\exp\left(\int p(x)dx\right)dx+C \qquad , \tag{2.89}$$

where C is a Real constant. Hence

$$y(x)=\exp\left(-\int p(x)dx\right)\int q(x)\exp\left(\int p(x)dx\right)dx+C\exp\left(-\int p(x)dx\right). \tag{2.90}$$

The above procedure shows that the function $y(t)$ given by eqn. (2.90) constitutes the general solution of the eqn. (2.25). The following example illustrate usefulness of the method of integrating factor.

Example 2.8.

Find the particular solution of the equation

$$y'+2xy=x \quad , \tag{2.91}$$

which satisfies the condition $y(0) = 1$.

In the eqn. (2.91) $p(x)=2x$, then the integrating factor equals to

$$\exp\left(\int p(x)dx\right)=\exp\left(\int 2xdx\right)=\exp\left(x^2\right) \quad . \tag{2.92}$$

Multiplying both sides of (2.91) by (2.92) the following equation yields

$$\exp\left(x^2\right)y'+\exp\left(x^2\right)2xy=x\exp\left(x^2\right) \quad , \tag{2.93}$$

which can be rewritten as

$$\left(\exp\left(x^2\right)y\right)'=x\exp\left(x^2\right). \tag{2.94}$$

The function on the right hand side of the equation (2.94) can be integrated by substitution (see Goldmann ….). Namely let $z=x^2$ then $dz=2xdx$ and hence

$$\int x\exp\left(x^2\right)dx=\frac{1}{2}\int\exp(z)dz=\frac{1}{2}\exp(z)=\frac{1}{2}\exp\left(x^2\right) \quad . \tag{2.95}$$

Consequently, after integrating both sides of eqn. (2.94) we obtain the following equation

$$\exp\left(x^2\right)y=\frac{1}{2}\exp\left(x^2\right)+C , \tag{2.96}$$

where C is a real constant. This implies that the general solution of the equation (2.91) takes form

$$y=\frac{1}{2}+C\exp\left(-x^2\right). \tag{2.97}$$

Using the initial condition $y(0) = 1$ the constant C can be easily determined as

$1 = y(0) = \frac{1}{2} + C \quad \Rightarrow \quad C = \frac{1}{2}$, which gives the final for of the particular solution under consideration as

$$y = \frac{1}{2} + \frac{1}{2}\exp\left(-x^2\right). \tag{2.98}$$

Chapter 3
Linear ordinary differential equations of higher order

3.1. Homogeneous linear differential equations

Assume that $p_1(t), p_2(t),..., p_{n-1}(t), p_n(t)$ are continuous function on given interval (a,b). The ordinary differential equation of the *n*th order of the type

$$y^{(n)} + p_1(t)y^{(n-1)} + p_2(t)y^{(n-2)} + ... + p_{n-1}(t)y' + p_n(t)y = q(t) \tag{3.1}$$

is called a ***linear ordinary differential equation of the order n (lde_n)***.

Before introducing some basic methods in ordinary differential equations of the higher order we formulate the existence and uniqueness theorem.

Theorem 3.1.

Let us consider a linear differential equation of the form (3.1). Assume that functions $p_1(t), p_2(t),..., p_n(t)$ are continuous on the interval (a, b). Then for any point $(t_0, y_0, y_1,..., y_{n-1}) \in (a,b)\times \mathbf{R^n}$ the Cauchy problem:

$$y^{(n)} + p_1(t)y^{(n-1)} + p_2(t)y^{(n-2)} + ... + p_{n-1}(t)y' + p_n(t)y = q(t) \tag{3.2}$$

and $\quad y(t_0) = y_0, \quad y'(t_0) = y_1, ... \, y^{(n-1)}(t_0) = y_{n-1}$

has exactly one solution in the interval (a, b).

The proof of this theorem can be found in monographs concerning ordinary differential equations.

In most cases, similarly to the first order case, a way to solve a linear equation of the form (3.1) leads through solving the following ***linear homogeneous ordinary differential equation of the order n (lhde_n)*** equation:

$$y^{(n)} + p_1(t)y^{(n-1)} + p_2(t)y^{(n-2)} + ... + p_{n-1}(t)y' + p_n(t)y = 0 \quad . \tag{3.3}$$

Fact 3.1.. Assume that $y_1(t)$ and $y_2(t)$ are two solutions of the equation (3.3) and *α*, *β* are two arbitrary real numbers. Then $y_3(t) = \alpha\, y_1(t) + \beta\, y_2(t)$ is also a solution of equation (3.3).

Proof. Due o linearity of differentiation following equalities hold:

$$\begin{aligned}
&y_3^{(n)} + p_1(t)y_3^{(n-1)} + p_2(t)y_3^{(n-2)} + ... + p_{n-1}(t)y_3^{'} + p_n(t)y_3 = (\alpha y_1 + \beta y_2)^{(n)} + p_1(t)(\alpha y_1 + \beta y_2)^{(n-1)} + \\
&+ p_2(t)(\alpha y_1 + \beta y_2)^{(n-2)} + ... + p_{n-1}(t)(\alpha y_1 + \beta y_2)^{'} + p_n(t)(\alpha y_1 + \beta y_2) = \alpha y_1^{(n)} + \beta y_2^{(n)} + \alpha p_1(t)y_1^{(n-1)} + \\
&+ \beta p_1(t)y_2^{(n-1)} + \alpha p_1(t)y_1^{(n-1)} + \alpha p_2(t)y_1^{(n-2)} + \beta p_2(t)y_2^{(n-2)} + ... + \alpha p_{n-1}(t)y_1^{'} + \beta p_{n-1}(t)y_2^{'} + \alpha p_n(t)y_1 + \\
&+ \beta p_n(t)y_2 = \alpha\left(y_1^{(n)} + p_1(t)y_1^{(n-1)} + p_2(t)y_1^{(n-2)} + ... + p_{n-1}(t)y_1^{'} + p_n(t)y_1\right) + \\
&+ \beta\left(y_2^{(n)} + p_1(t)y_2^{(n-1)} + p_2(t)y_2^{(n-2)} + ... + p_{n-1}(t)y_2^{'} + p_n(t)y_2\right)
\end{aligned} \tag{3.4}$$

Using assumption that $y_1(t)$ and $y_2(t)$ are solutions of the equation (3.3) we can write

$$\alpha\left(y_1^{(n)} + p_1(t)y_1^{(n-1)} + p_2(t)y_1^{(n-2)} + \ldots + p_{n-1}(t)y_1^{'} + p_n(t)y_1\right) +$$
$$+\beta\left(y_2^{(n)} + p_1(t)y_2^{(n-1)} + p_2(t)y_2^{(n-2)} + \ldots + p_{n-1}(t)y_2^{'} + p_n(t)y_2\right) = 0 \,. \qquad (3.5)$$

The eqs. (3.4) and (3.5) show that $y_3(t)$ is a solution (3.3), which completes the proof.
Corollary 3.1. The set of solutions of a linear homogeneous differential equation constitute a linear space.
A general solution of the equation (3.3) bases on so-called **fundamental set of solutions**.
Definition 3.1. Let $y_1(t), y_2(t),..., y_n(t)$ be a set of solutions to the equation of (3.3) defined on the interval (a, b). The set $y_1(t), y_2(t),..., y_n(t)$ is called a fundamental system of solutions to the eqn. (3.3) on the interval (a, b) if for any $t \in (a,b)$ the following condition holds

$$\det\begin{bmatrix} y_1(t) & y_2(t) & \ldots & y_n(t) \\ y_1^{'}(t) & y_2^{'}(t) & \ldots & y_n^{'}(t) \\ \ldots. & & & \\ y_1^{(n-1)}(t) & y_2^{(n-1)}(t) & \ldots & y_n^{(n-1)}(t) \end{bmatrix} \neq 0 \qquad . \qquad (3.6)$$

The determinant in the equation above is called the **Wronskian** of the eqn. (3.3) and usually denoted by $W(t) = W(y_1(t), y_2(t),..., y_n(t))$.
Example 3.1. Consider the linear homogeneous equation of the second order

$$2t^2 y'' + 3ty' - y = 0 \qquad (3.7)$$

on the open interval $(0,\infty)$. Let us check that following pair of functions

$$y_1(t) = \sqrt{t} \quad ; \quad y_2(t) = \frac{1}{t} \qquad (3.8)$$

is a fundamental set of solutions for eqn. (3.7). We begin from computing derivatives of functions y_1 and y_2

$$y'(t) = \frac{1}{2}\cdot\frac{1}{\sqrt{t}} \qquad y_2'(t) = \frac{(-1)}{t^2} \qquad . \qquad (3.9)$$

Hence the Wronskian corresponding to eqn (3.7) is

$$\det\begin{bmatrix} \sqrt{t} \;\; ; & \frac{1}{t} \\ \frac{1}{2}\frac{1}{\sqrt{t}} \;\; ; & \frac{(-1)}{t^2} \end{bmatrix} = \frac{(-1)}{t^2} \quad \sqrt{t} - \frac{1}{t}\cdot\frac{1}{2}\frac{1}{\sqrt{t}} = -\frac{\sqrt{t}}{t^{\frac{3}{2}}} - \frac{1}{2t\sqrt{t}} = -\frac{3}{2t\sqrt{t}} =$$
$$= -\frac{2\sqrt{t}+\sqrt{t}}{2t^2} \neq 0 \qquad dla \quad t \notin (0;\infty) \qquad (3.10)$$

His proves that the set of functions (3.9) is a fundamental set of solutions of eqn. (3.7).

One of the most important theorems concerning Wronskian and finding a fundamental set of solutions is the Liouville theorem.

Theorem 3.2 (Liouville's formula)

Let $y_1(t), y_2(t),..., y_n(t)$ be a fundamental set of solutions of a linear homogeneous differential equation (3.3) defined on an interval (a, b) and $t_0 \in (a,b)$. Then for any $t \in (a,b)$ Wronskian $W(t) = W(y_1(t), y_2(t),..., y_n(t))$ satisfies the following condition:

$$W(t) = W(t_0)\exp\left(-\int_{t_0}^{t} p_1(\tau)d\tau \right) \quad . \tag{3.11}$$

Proof of this theorem can be found in monographs concerning ordinary differential equations. The next example demonstrates usefulness of the above formula in receiving members of a fundamental set of solutions.

Example 3.2. Consider the following lhde_2:

$$y'' + \frac{1-2t}{t}y' + \frac{t-1}{t}y = 0 \quad , \tag{3.12}$$

defined for $t > 0$. By direct substitution it is easy to verify that the function $y_1(t) = e^t$ satisfies the eqn. (3.12). Now we apply the Liuville's formula to receive another solution y_2 such that y_1 and y_2 constitute the fundamental set of solutions to eqn. (3.12). Let $t_0 \in (0,+\infty)$. Then due to Liouville's formula

$$\det\begin{bmatrix} y_1(t), & y_2(t) \\ y_1'(t), & y_2'(t) \end{bmatrix} = \det\begin{bmatrix} y_1(t_0), & y_2(t_0) \\ y_1'(t_0), & y_2'(t_0) \end{bmatrix}\exp\left(-\int_{t_0}^{t} p_1(\tau)d\tau \right) \tag{3.13}$$

Assume now that $t_0 = 1$. Due to theorem 3.1. there exists only one solution satisfying the conditions: $y_2(1) = 0, \quad y_2'(1) = 1$. Then by Liouville's formula

$$\det\begin{bmatrix} e^t, & y_2(t) \\ e^t, & y_2'(t) \end{bmatrix} = \det\begin{bmatrix} e, & 0 \\ e, & 1 \end{bmatrix} e^{-\int_1^t \frac{1-2\tau}{\tau}dt} = e \cdot e^{-\left[\int_1^t \frac{1}{\tau} - 2\int_1^t dt\right]} = e \cdot e^{-\left[e_n|\tau|\Big|_1^t - 2\tau\Big|_1^t\right]} = e \cdot e^{-[\ln t - [2t-2]]} = \\ = e \cdot e^{\ln\frac{1}{t}} \cdot e^{2t} \cdot e^{-2} = e^{-1} \cdot \frac{1}{t}e^{2t} = \frac{1}{t}e^{2t-1} \quad . \tag{3.14}$$

On the other hand

$$\det\begin{bmatrix} e^t, & y_2(t\ ) \\ e^t, & y_2'(t\ ) \end{bmatrix} = y'_2(t)e^t - e^t y_2(t) \tag{3.15}$$

Comparing (3.14) and (3.15) one gets

$$y'e^t - e^t y = \frac{1}{t}e^{2t-1} \,/ : e^t \quad \Rightarrow \quad y' - y = \frac{1}{t}e^{t-1} \quad . \tag{3.16}$$

The last equation in (3.16) is a linear ordinary differential equation of the first order with associated initial condition $y_2(1)=0$. According to the equation (2.30) the solution of the corresponding homogeneous equation is

$$y(t)=C\cdot e^{-\int(-1)dt}=Ce^t \tag{3.17}$$

In order to find general solution to eqn. (3.16) let us apply the constant variation method.

$$\begin{array}{l} y=C(t)\cdot e^t \quad \Rightarrow \quad y'=C'(t)\cdot e^t+C(t)e^t \quad \Rightarrow \quad C'(t)e^t+C(t)e^t-C(t)\cdot e^t=\frac{1}{t}e^{t-1} \\ \Rightarrow \quad C'(t)e^t=\frac{1}{t}e^{t-1}/:e^t \quad \Rightarrow \quad C'(t)=e^{-1}\cdot\frac{1}{t} \quad \Rightarrow \quad C(t)=\frac{1}{e}\int\frac{1}{t}dt \end{array}. \tag{3.18}$$

Because $t\in(0,+\infty)$ the function $C(t)$ in (3.18) can be expressed as

$$C(t)=e^{-1}\cdot\ln t+C. \tag{3.19}$$

Therefore the solution of eqn. (3.16) is given by

$$y(t)=\left[e^{-1}\ln t+C\right]\cdot e^t=e^{t-1}\ln t+Ce^t. \tag{3.20}$$

Utilising the initial condition $y_2(1)=0$ one gets

$$y(1)=0\Rightarrow \quad e^{1-1}\ln 1+Ce^1=0 \quad \Rightarrow C=e^{-1} \quad \Rightarrow y(t)=e^{t-1}\cdot\ln t-e^{-1}\cdot e^t=e^{t-1}\ln t. \tag{3.21}$$

It is easy to see that due to (3.16) the pair of functions

$$(e^t, e^{t-1}\ln t) \tag{3.22}$$

satisfies definition 3.1 and therefore constitutes the fundamental set of solutions to the equation (3.12).

It has been already mentioned that solutions of any lhde_n constitute a linear space. Let us now turn to some algebraic properties of solutions. The first one is reminding the concept of linear independence.

Definition 3.2. Functions $y_1(t), y_2(t),..., y_n(t)$ are linearly independent on the interval (*a*, *b*) if and only if for any set of real constants $\alpha_1, \alpha_2,...,\alpha_n$ the relation

$$\forall t\in(a,b) \quad \alpha_1 y_1(t)+\alpha_2 y_2(t)+...+\alpha_n y_n(t)=0 \tag{3.23}$$

Implies that

$$\alpha_1=\alpha_{21}=...=\alpha_n=0. \tag{3.24}$$

In the opposite case the functions are called to be linearly dependent.

It can be proved that the following fact holds true.

Fact 3.2. Let functions $y_1(t), y_2(t),..., y_n(t)$ be differentiable on an open interval (*a*, *b*). Then

a) If $W(y_1(t), y_2(t),..., y_n(t))\neq 0$ for any $t_0\in(a,b)$, then functions $y_1(t), y_2(t),..., y_n(t)$ are linearly independent on (*a*, *b*).

b) If functions $y_1(t), y_2(t),..., y_n(t)$ are linearly dependent on (*a*, *b*), then

$$\forall t \in (a,b) \quad W(y_1(t), y_2(t),..., y_n(t)) = 0 \quad . \tag{3.25}$$

If we sum up the all given above properties of fundamental set of solutions of a lhde_n one can see that fundamental set of solutions is a set of n linearly independent functions in the linear space of solutions of given lhde_n. Moreover, it can be proved that the dimension of this linear space is *n*. This immediately gives the fact

Fact 3.3. The fundamental set of solutions of a given linear homogeneous differential equation is an algebraic base in the linear space of solutions of this equation.

It is well-known, however, that having a base of given linear space one can obtain any vector of this space as a linear combination of the elements of the base. This way we have damonstarted that the following theorem holds true.

Theorem 3.3. Let $y_1(t), y_2(t),..., y_n(t)$ be a fundamental set of solutions of given lhde_n. Then the general solution of this equation is given by the following formula

$$y(t) = C_1 y_1(t) + C_2 y_2(t) + ... + C_n y_n(t) \quad , \tag{3.26}$$

Where C_1, C_2,…,C_n are real constant.

The usefulness of this very important theorem is demonstrated on the next example.

Example 3.3 Consider the lhd2_2

$$y'' + \frac{1-2t}{t} y' + \frac{t-1}{t} y = 0 \tag{3.27}$$

for $t > 0$. Find the particular solution satisfying initial conditions $y(1) = e$ and $y'(1) = 2e$.

In the Example 3.2. we have found a fundamental set of solutions of the eqn. (3.27) in the form given by. Then by virtue of the theorem 3.3 we can write the general solution of (3.27) as

$$y(t) = C_1 e^t + C_2 e^{t-1} \ln t \quad . \tag{3.28}$$

In order to solve the problem let us evaluate the first derivative of the function *y* in the eqn. (3.28).

$$y'(t) = C_1 e^t + C_2 \left(e^{t-1} \ln t + e^{t-1} \frac{1}{t} \right) \quad . \tag{3.29}$$

By the use of the initial conditions one gets the following system of equations

$$e = C_1 e^1 + C_2 e^0 \ln 1 \quad and \quad 2e = C_1 e^1 + C_2 \left(e^0 \ln 1 + e^0 \cdot 1 \right) \ . \tag{3.30}$$

Solving it one obtains: $C_1 = 1$ and $C_2 = e$. Hence the particular solution that we are looking for is the following function:

$$y_1 = e^t + e^t \ln t \, . \tag{3.31}$$

The theorem 3.3 guarantees that if we know a fundamental set of solution of lhde_n under consideration then we know its general solution. However, as it is always the case of algebraic base, the fundamental set of solutions associated with a given lhde_n is not a unique one.

Another set can be obtained by appropriate linear transformation. On the other hand if we now any set of functions satisfying the condition (3.6) then the corresponding lhde_n is given in a unique way. This fact demonstrates the following example.

Example 3.4.

Find a lhde_2 that has the following fundamental set of solutions.

$$y_1(t) = t^3 \qquad y_2(t) = t^4 \quad . \tag{3.32}$$

Due to theorem 3.3 the general solution has the form

$$y(t) = C_1 t^3 + C_2 t^4 \tag{3.33}$$

and the first derivative is

$$y'(t) = 3C_1 t^2 + 4C_2 t^3 \quad . \tag{3.34}$$

Now consider equations (3.33) and (3.34) as a system and let us solve this system with respect to unknown C_1 and C_2.

$$\begin{aligned} &C_1 = \frac{y - C_2 t^4}{t^3} = \frac{1}{t} y - C_2 \cdot t \quad \Rightarrow \quad y' = 3 \cdot (\frac{1}{t^3} y - C_2 t) \cdot t^2 + 4C_2 \cdot t^3 \\ &y' = \frac{3}{t} y - 3C_2 t^3 + 4C_2 t^3 = \frac{3}{t} y + C_2 t^3 \quad \Rightarrow \quad C_2 = \frac{y' - \frac{3}{t} y}{t^3} = \frac{1}{t^3} y' - \frac{3}{t^4} y \end{aligned} \tag{3.35}$$

$$C_1 = \frac{1}{t^3} y - (\frac{1}{t^3} y' - \frac{3}{t^4} y) \cdot t - \frac{1}{t^2} y' + \frac{3}{t^3} y \quad = \quad \frac{4}{t^3} y - \frac{1}{t^2} y' \tag{3.36}$$

The second derivative of the general solution (3.33) is

$$y''(t) = 6C_1 t + 12C_2 t^2 \quad . \tag{3.37}$$

Substituting C_1 given by (3.35) and C_2 given by (3.36) to eqn. (3.37) gives

$$y'' = 6 \cdot (\frac{4}{t^3} y - \frac{1}{t^2} y') \cdot t + 12(\frac{1}{t^3} y' - \frac{3}{t^4} y) \cdot t^2 \quad = \quad \frac{24}{t^2} y - \frac{6}{t} y' + \frac{12}{t} y' - \frac{36}{t^2} y = \frac{6}{t} y' - \frac{12}{t^2} y \tag{3.38}$$

Finally the equation generated by the fundamental set of solution (3.32) has the following form

$$y'' - \frac{6}{t} y' + \frac{12}{t^2} y = 0 \; . \tag{3.39}$$

Now we demonstrate that if a particular solution of a lhde_n is known then by using this solution it is possible to reduce by one the order of the lhde_n.

Theorem 3.4. Assume that $\varphi(t)$ is a nontrivial particular solution of lhde_n then by substitution

$$y = \varphi(t) \int z dt \tag{3.40}$$

the lhde_n under consideration can be reduced to a lhde_n-1.

Proof. The proof will be given for the second order linear homogeneous equation. Let $\varphi(t)$ be a nontrivial particular solution of the equation

$$y''+p_1(t)y'+p_2(t)y=0\,. \tag{3.41}$$

First two derivatives of the function (3.40) are given by

$$y'=\varphi'(t)\int z(t)dt+\varphi(t)z(t) \tag{3.42}$$

$$y''=\varphi''(t)\int z(t)dt+\varphi'(t)z(t)+\varphi'(t)z(t)+\varphi(t)z'(t) \tag{3.43}$$

Substituting (3.42) and (3.43) to (3.41) one gets

$$\begin{aligned}&\left|\varphi''(t)\int z(t)dt+2\varphi'(t)z(t)+\varphi(t)z'(t)\right|+p_1(t)\left|\varphi'(t)\int z(t)dt+\varphi(t)z(t)\right|+\\&+p_2(t)\left[\varphi(t)\int z(t)dt\right]=0\end{aligned} \tag{3.44}$$

$$\left[\varphi''(t)+p_1(t)\varphi'(t)+p_2(t)\varphi(t)\right]\int z(t)dt+2\varphi'(t)z(t)+\varphi(t)z'(t)+p_1(t)\varphi(t)z(t)=0 \tag{3.45}$$

The first term in eqn. (3.45) equals zero because $\varphi(t)$ is a solution of (3.41). Therefore

$$2\varphi'(t)z(t)+\varphi(t)z'(t)+p_1(t)\varphi(t)z(t)=0\qquad. \tag{3.46}$$

As $\varphi(t)$ is a nontrivial solution of (3.41) then dividing both sides of (3.46) by $\varphi(t)$ leads to

$$z'(t)+z(t)\left[2\varphi'(t)+p_1(t)\varphi(t)\right]=0\qquad. \tag{3.47}$$

Equation (3.47) is a linear homogeneous differential equation of the first order, which completes the proof.

The proof of the above theorem in the case of n-th order equation can be carried out analogically. Now let us demonstrate an example of an application of the theorem 3.4.

Example 3.5.

Find the general solution to the following equation

$$y''+\frac{2t}{t^2-1}y'-\frac{2}{t^2-1}y=0 \tag{3.48}$$

in the interval $(1,+\infty)$. It easy to see that function $\varphi(t)=t$ is a nontrivial solution of (3.17), then according to theorem 3.4 substitution

$$y=t\int zdt \tag{3.49}$$

has to lead to reduction of eqn. (3.49) to a linear homogeneous equation of the first order. The first and the second derivatives of the function given by eqn. (3.49) are given by

$$y' = \int z dt + tz\,; \qquad y'' = z + z + t\cdot z' = 2z + tz' \quad . \tag{3.50}$$

Substitution (3.49) and (3.50) to (3.51) gives

$$2z + tz' + \frac{2t}{t^2-1}\left(\int z dt + tz\right) - \frac{2t}{t^2-1}\int z dt = 0\,. \tag{3.51}$$

By reducing terms we obtain

$$tz' + 2z\left(1 + \frac{t^2}{t^2-1}\right) = 0\,. \tag{3.52}$$

Because $t \in (1,+\infty)$ then dividing both sides by t eqn. (3.52) takes the form

$$z' + 2\frac{2t^2-1}{t^3-t} z = 0\,. \tag{3.53}$$

Due to theorem 2.2 the solution of the above equation can be obtained by means of the formula

$$z(t) = C\cdot e^{-\int \frac{2t^2-1}{t^3-1}dt} \tag{3.54}$$

The integral in the exponent can be evaluated as follows

$$\begin{aligned}&\int 2\frac{2t^2-1}{t^3-1}dt = 2\int\left\{\frac{3t^2-1}{t^3-1} - \frac{t^2}{t^3-t}\right\}dt = 2\left[\ln\left|t^3-t\right| - \int\frac{tdt}{t^2-1}\right] = \ln\left(t^3-t\right)^2 - \int\frac{2t}{t^2-1}dt = \\ &= \ln\left(t^3-t\right)^2 - \ln\left(t^2-1\right) = \ln\frac{\left(t^3-t\right)^2}{\left(t^2-1\right)} = \ln\frac{t\left(t^2-1\right)\cdot t\left(t^2-1\right)}{t^2-1} = \ln t^2\left(t^2-1\right)\end{aligned} \quad . \tag{3.55}$$

Hence

$$z(t) = C\cdot\frac{1}{t^2\left(t^2-1\right)} \tag{3.56}$$

Now the solution can be obtained by substitution (3.56) into (3.49)

$$\begin{aligned}&y(t) = t\int\frac{C}{t^2\left(t^2-1\right)}dt = Ct\int\left(\frac{1}{t^2-1} - \frac{1}{t^2}\right)dt = Ct\left[\frac{1}{2}\left(\int\frac{dt}{t-1} - \int\frac{dt}{t+1}\right) + \frac{1}{t}\right] = \\ &= Ct\left[\frac{1}{2}\left(\ln(t-1) - \ln(t+1)\right) + C_1 + \frac{1}{t}\right] = C\frac{t}{2}\ln\frac{t-1}{t+1} + C_2 t + C = C\left(\frac{t}{2}\ln\frac{t-1}{t+1} + 1\right) + C_2 t\end{aligned} \quad . \tag{3.57}$$

We have already shown that any solution of given lhde_n can be obtained if its fundamental set of solutions is known. Unfortunately in general case there is no universal procedure that determines the fundamental set of solutions. There is, however, a special case,

when such a procedure can be formulated. We will orient our consideration towards this case now.
Let us consider a linear homogeneous differential equation of the form

$$y^{(n)} + p_1 y^{(n-1)} + p_2 y^{(n-2)} + \dots + p_{n-1} y' + p_n y = 0 \qquad , \qquad (3.58)$$

Where p_1, p_2,…, p_n are real constant. The above equation is called **linear homogeneous differential equation of constant coefficients of order n** (lhdecc_n)
In order to solve an equation of the type (3.58) it is necessary to consider the **characteristic polynomial**, which corresponds to given equation. The characteristic polynomial, which correspond to the equation (3.58) takes the following form:

$$p(\lambda) = \lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \dots + p_{n-1}\lambda + p_n \qquad . \qquad (3.59)$$

Note that the corresponding coefficients of lhdecc_n and its characteristic polynomial are the same. The following theorem shows how a fundamental set of solutions of eqn. (3.58) can be obtained.

Theorem 3.5. (Fundamental set of solutions to linear differential equation of nth order with constant coefficients).

Consider an equation of the form (3.59). Assume that λ_1, λ_2,…, λ_s are real roots of corresponding to eqn. (3.59) characteristic polynomial with multiplicities equal to k_1, k_2,…, k_s, respectively. Let $\lambda_{s+1} = \alpha_1 + i\beta_1$, $\overline{\lambda}_{s+1} = \alpha_1 - i\beta_1, \dots, \lambda_{s+m} = \alpha_m + i\beta_m$, $\overline{\lambda}_{s+m} = \alpha_m - i\beta_m$, where $\beta_1 \neq 0, \beta_2 \neq 0, \dots, \beta_m \neq 0$, be complex roots with multiplicities $l_1, l_2, \dots, l_m$, respectively, of characteristic polynomial $p(\lambda)$ corresponding to the eqn. (3.59), provided that $k_1 + k_2 + \dots + k_s + 2(l_1 + l_2 + \dots + l_m) = n$. Then the fundamental set of solutions to eqn. (3.59) consists of the following functions:

$$\begin{array}{l} e^{\lambda_1 t}, te^{\lambda_1 t}, \dots \ t^{k_1-1}e^{\lambda_1 t} \\ \dots\dots\dots\dots\dots\dots\dots \\ e^{\lambda_s t},\ te^{\lambda_s t}, \dots \ t^{k_s-1}e^{\lambda_s t} \end{array} \qquad (3.60)$$

$$\begin{array}{l} \begin{cases} e^{\alpha_1 t}\cos\beta_1 t \\ e^{\alpha_1 t}\sin\beta_1 t \end{cases}, \begin{cases} te^{\alpha_1 t}\cos\beta_1 t \\ te^{\alpha_2 t}\sin\beta_1 t \end{cases}, \begin{cases} t^{l_1-1}e^{\alpha_1 t}\cos\beta_1 t \\ t^{l_1-1}e^{\alpha_2 t}\sin\beta_1 t \end{cases}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ \begin{cases} e^{\alpha_m t}\cos\beta_m t \\ e^{\alpha_m t}\sin\beta_n t \end{cases}, \begin{cases} te^{\alpha_m t}\cos\beta_m t \\ te^{\alpha_m t}\sin\beta_m t \end{cases}, \begin{cases} t^{lm-1}e^{\alpha_m t}\cos\beta_m t \\ t^{l_m-1}e^{\alpha_m t}\sin\beta_m t \end{cases} \quad . \end{array} \qquad (3.61)$$

Remarks.

1. A root λ_0 of a polynomial $p(\lambda)$ is called to be of multiplicity k if and only if $p(\lambda)$ is dividable by the expression $(\lambda - \lambda_0)^k$.
2. Please note that $p(\lambda)$ is a polynomial whose coefficients are real numbers, therefore if any complex number λ_0 is a root of $p(\lambda)$ then the conjugate number $\overline{\lambda}_0$ is a root of $p(\lambda)$ as well.

Now some illustrative examples concerning application of the above theorem will be presented.
Example 3.5. Find the general solution to the following lhdfcc_4:

$$2y^{(4)} - 5y''' + 5y' - 2y = 0 \quad . \tag{3.62}$$

The corresponding characteristic polynomial is

$$p(\lambda) = 2\lambda^4 - 5\lambda^3 + 5\lambda - 2 \quad . \tag{3.63}$$

It can be factorised in the following way

$$\begin{aligned} p(\lambda) &= 2\lambda^4 - 5\lambda^3 + 5\lambda - 2 = 2(\lambda^4 - 1) - 5(\lambda^3 - \lambda) = 2(\lambda^2 - 1)(\lambda^2 + 1) - 5\lambda(\lambda^2 - 1) = \\ &= (\lambda^2 - 1)(2(\lambda^2 + 1) - 5\lambda) = (\lambda - 1)(\lambda + 1)(2\lambda^2 - 5\lambda + 2) = (\lambda - 1)(\lambda + 1)(\lambda - 2)\left(\lambda - \frac{1}{2}\right) \end{aligned} \quad . \tag{3.64}$$

It means that $p(\lambda)$ has four different real roots: $\lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = 2, \quad \lambda_4 = \frac{1}{2}$.

Then by virtue of the theorem 3.4 the fundamental set of solutions of eqn. (3.62) is the following one:

$$y_1(t) = e^t , \quad y_2(t) = e^{-t} , \quad y_3(t) = e^{2t} \quad y_4(t) = e^{\frac{t}{2}} \quad . \tag{3.65}$$

Hence the general solution is found by applying theorem 3.3 as

$$y = C_1 e^t + C_2 e^{-t} + C_3 e^{2t} + C_4 e^{\frac{t}{2}} . \tag{3.66}$$

The next example will be a case where complex roots appear.
Example 3.6.
Find the general solution to the following lhdfcc_6

$$y^{(6)} + y^{(4)} - 2y'' = 0 . \tag{3.67}$$

Equating the corresponding characteristic polynomial to zero one gets

$$\lambda^6 + \lambda^4 - 2\lambda^2 = 0 \quad \Leftrightarrow \quad \lambda^2(\lambda^4 + \lambda^2 - 2) = 0 \quad . \tag{3.68}$$

This means that $\lambda_1 = \lambda_2 = 0$. The second equation

$$\lambda^4 + \lambda^2 - 2 = 0 \tag{3.69}$$

factor in the last equation is a biquadratic. By substitution $\lambda^2 = t$ one gets the quadratic equation

$$t^2 + t - 2 = 0 \quad , \tag{3.70}$$

that has the roots $t_1 = -2$ and $t_2 = 1$. By taking square roots of them one obtains the next four roots of the eqn. (3.68), with two complex numbers among them: $\lambda_1 = 0, \lambda_2 = 0$, $\lambda_3 = -1$, $\lambda_4 = 1$, $\lambda_5 = -\sqrt{2}i$, $\lambda_6 = \sqrt{2}i$. Please note that, as it was indicated in the Remark 2, the roots λ_5 and λ_6 are mutually conjugated complex numbers. Then according Theorem 3.4 the fundamental set of solutions of eqn. (3.66) consists of the following functions:

$$y_1(t) = 1 \;;\; y_2(t) = t \;;\; y_3(t) = e^{-t} \;;\; y_4(t) = e^t \;;\; y_5(t) = \cos\sqrt{2}t \;;\; y_6(t) = \sin\sqrt{2}t \quad . \tag{3.71}$$

Finally the general solution of the eqn. (3.67) takes the following form:

$$y = C_1 + C_2 t + C_3 e^{-t} + C_4 e^t + C_5 \cos\sqrt{2}t + C_6 \sin\sqrt{2}t \;. \tag{3.72}$$

Let us now tern to initial and boundary value problems.
Example 3.7. Find the particular solution of the equation

$$y''' - 3y'' + 3y' - y = 0 \quad , \tag{3.73}$$

which satisfies the following conditions: $y(0) = 1$, $y'(0) = 2$ and $y''(0) = 2$.

The characteristic polynomial

$$p(\lambda) = \lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3 \tag{3.74}$$

has the triple root: $\lambda_1 = \lambda_2 = \lambda_3 = 1$. Hence the fundamental set of solutions consists of three following functions

$$y_1(t) = e^t \;; \quad y_2(t) = te^t \;; \quad y_3(t) = t^2 e^t \tag{3.75}$$

and the general solution is

$$y(t) = C_1 e^t + C_2 te^t + C_3 t^2 e^t \quad . \tag{3.76}$$

Now let us use the initial conditions to determine the constants.
The first condition leads to $y(0) = 1 \Rightarrow C_1 = 1$. The first derivative of function y in eqn. (3.76) is

$$y'(t) = C_1 e^t + C_2\left(e^t + te^t\right) + C_3\left(2te^t + t^2 e^t\right) \quad . \tag{3.77}$$

Then the second condition gives $y'(0) = 2 \quad \Rightarrow \quad y'(0) = C_1 + C_2(1+0) + C_3(0+0) = 2 \quad \Rightarrow$
$\Rightarrow C_1 + C_2 = 2$. Because $C_1 = 1$, hence $C_2 = 1$. Now let us compute the second derivative

$$y''(t) = C_1 e^t + C_2 e^t + C_2\left(e^t + te^t\right) + 2C_3\left(e^t + te^t\right) + C_3\left(2te^t + t^2 e^t\right) \tag{3.78}$$

Finally applying the third condition one gets
$y''(0) = 2 \quad \Rightarrow y''(0) = C_1 + C_2 + C_2 + 2C_3 = 2 \quad \Rightarrow \quad C_1 + 2C_2 + 2C_3 = 2$.

Because $C_1 = C_2 = 1$ one gets $3 + 2C_3 = 2 \Rightarrow C_3 = -\frac{1}{2}$. Substituting the constants C_1, C_2 and C_3 into (3.78) we obtain the required particular solution

$$y(t) = e^t + te^t - \frac{1}{2}\ t^2 e^t \quad . \tag{3.79}$$

The last example deals with boundary conditions.
Example 3.8. One-dimensional boundary value problems
Find a particular solutions of the equation

$$y'' + y = 0, \tag{3.80}$$

that satisfy the following boundary conditions:
a.) y(0) = 0 and y(π/2) = 1 (first problem); b.) y(0) = 0 and y(π) = 0 (second problem); c.) y(-π) = -1 and y(π) = 1 (third problem). The equation (3.80) is a linear homogeneous equation of the second order. Its characteristic polynomial is $p(\lambda) = \lambda^2 + 1$, with roots $\lambda_1 = i$ and $\lambda_2 = -i$. Then according to the Theorem 3.4 the fundamental set of solutions to the eqn. (3.80) consists of the functions: y_1(t) = cos*t* and y_2(t) = sin*t* . Consequently the general solution of (3.80) takes the form:

$$y(t) = C_1 \sin t + C_2 \cos t \ . \tag{3.81}$$

Consider now the conditions of case a.). Substituting them to the eqn. (3.81) one gets $C_1 = 1$ and $C_2 = 0$. Then the particular solution is

$$y(t) = \sin t \tag{3.82}$$

In this case there exists a unique solution to the boundary value problem under consideration. In the case b.), however, substitution boundary conditions into (3.81) leads to $C_2 = 0$, but C_1 can take any real value. It means the solution exists, but it is not unique. Moreover, there are a lot of solutions to the boundary value problem b.). Finally let us consider the case c.). Substitution of boundary conditions into (3.81) leads to contradiction $C_2 = 1$ and $C_2 = -1$ simultaneously, which constitute a contradiction. In this case a solution to the boundary problem does not exist. Cases b.) and c.) are examples of so-called ill-posed or ill-conditioned problems. These cases will be commented in section 4.6.

3.2. Non-homogeneous ordinary differential equations

In the previous section only linear homogeneous differential equation were considered. Now let us gives some ideas concerning equations that are not homogeneous. At first we have to note that Theorem 2.4 formulated in Chapter 2 remains true for linear differential equations of the higher order. Now it can be stated as follows
Theorem 3.6.
Consider lde_n of the form (3.1). Assume that y_h is the general solution of the corresponding homogeneous equation (3.3) and y_p is any particular solution of the corresponding lhde_n in the form (3.1). Then the general solution *y* to the equation (3.1) would be

$$y = y_h + y_p \quad . \tag{3.83}$$

The above theorem together with Theorem 3.3 allow to predict the general solution to the (3.1).

Corollary Let y_p be a particular solution to eqn. (3.1) and let $y_1(x), y_2(x), ..., y_n(x)$ be a fundamental set of solutions of corresponding to (3.1) homogeneous equation. Then the general solution of the (3.1) is given by

$$y(t) = C_1 y_1(x) + C_2 y_2(x) + ... + C_n y_n(x) + y_p \quad , \tag{3.84}$$

where $C_1, C_2, ..., C_n$ are real constant.

Now we consider the problem how a particular solution of (3.1) can be found. We discuss the method of constant variation (already known from the first order linear differential equations), which gives an universal tool for solving ordinary linear differential equations. The idea of the method will be explained for the case of the second order equation. Consider the following linear differential equation of the second order

$$y'' + p_1(x) y' + p_2(x) y = q(x) \quad . \tag{3.85}$$

Let $y_1(t), y_2(t)$ be a fundamental set of solutions of corresponding to (3.85) homogeneous equation. We will seek for a particular solution of (3.85) in the form

$$y(x) = C_1(x) y_1(x) + C_2(x) y_2(x) \tag{3.86}$$

Finding the first derivative of the above function gives

$$y'(x) = C_1'(x) \cdot y_1(x) + C_1(x) y_1'(x) + C_2'(x) y_2(x) C_2(x) \cdot y_2'(x). \tag{3.87}$$

Assume now that functions $C_1(x)$ and $C_2(x)$ are selected in such a way that satisfy the condition

$$C_1'(x) \cdot y_1(x) + C_2'(x) y_2(x) = 0. \tag{3.88}$$

Therefore eqn. (3.87) is reduced to

$$y'(x) = C_1(x) y_1'(x) + C_2(x) \cdot y_2'(x). \tag{3.89}$$

Now let us find the second derivative by means of the eqn. (3.89)

$$y''(x) = C_1'(x) y_1'(x) + C_1(x) y_1''(x) + C_2'(x) y_2'(x) + C_2(x) \cdot y_2''(x) \tag{3.90}$$

The second derivative (eqn. (3.90)), the firs one (eqn. (3.89)) and the function *y* (eqn. (3.86)) are now substituted into eqn. (3.81).

$$\begin{aligned} &C_1'(x) y_1'(x) + C_1(x) y_1''(x) + C_2'(x) y_2'(x) + C_2(x) \cdot y_2''(x) + \\ &+ p_1(x) \cdot [C_1(x) y_1'(x) + C_2(x) y_2(x)] + p_2(x) \cdot [C_1(x) y_1(x) + C_2(x) y_2(x)] = q(x) \end{aligned} \tag{3.91}$$

After some rearrangements one gets

$$C_1(x)[y_1''(x)+p_1(x)y_1'(x)+p_2(x)y_1(x)]+C_2(x)[y_2''(x)+p_1(x)y_2'(x)+p_2(x)y_2(x)]+ \\ +C_1'(x)y_1'(x)+C_2'(x)y_2'(x)=q(x) \quad . \qquad (3.92)$$

Because $y_1(x)$ and $y_2(x)$ are solutions of homogeneous linear equation corresponding to eqn. (3.85) then first two terms in eqn. (3.92) vanish. Then eqn. (3.92) reduces to

$$C_1'(x)y_1'(x)+C_2'(x)y_2'(x)=q(x) \qquad (3.93)$$

Consider now eqn. (3.88) and (3.93) as a system of two equations and solve them with respect $C_1'(x)$ and $C_2'(x)$. The solution can be written in the form

$$C_1'(x)=\frac{\det\begin{vmatrix}0 & y_2(x)\\ q(x) & y_2'(x)\end{vmatrix}}{\det\begin{vmatrix}y_1(x) & y_2(x)\\ y_1'(x) & y_2'(x)\end{vmatrix}} \quad ; \qquad C_2'(x)=\frac{\begin{vmatrix}y_1(x) & 0\\ y_1'(x) & q(x)\end{vmatrix}}{\det\begin{vmatrix}y_1(x) & y_2(x)\\ y_1'(x) & y_2'(x)\end{vmatrix}} \quad . \qquad (3.94)$$

Note that the denominators in both expressions in (3.94) are the same and are equal to Wronskian W(x) of the linear homogeneous equation corresponding to eqn. (3.85). Because the pair $y_1(t), y_2(t)$ is the fundamental set of solutions to (3.85) then Wronskian has non-zero values in the whole domain of the problem. Therefore the solution (3.94) always exists and the suggested method leads to solution. In order to find a particular solution it is necessary to integrate expressions in (3.94)

$$C_1(x)=\int\frac{\begin{vmatrix}0 & y_2(x)\\ q(x) & y_2'(x)\end{vmatrix}}{W(x)}dx+A; \qquad C_2(x)=\int\frac{\begin{vmatrix}y_1(x) & 0\\ y_1'(x) & q(x)\end{vmatrix}}{W(x)}dx+B, \qquad (3.95)$$

where *A*, *B* are real constant. Substitution $C_1(x)$ and $C_2(x)$ to eqn. (3.86) gives us the particular solution.

This way we have proved that suggested procedure allows to obtain a particular solution to a non-homogeneous linear equation if only a fundamental set of solutions to the corresponding homogeneous equation is known. Let us now illustrate the procedure by two examples.

Remark.

Note that in the case of the equation of the *n*-th order system of equations (3.88) and (3.93) has to be replaced by the following matrix equation

$$\begin{bmatrix} y_1(x) & y_2(x)\ldots.. & y_n(x)\\ y_1'(x) & y_2'(x)\ldots & y_n'(x)\\ . & & \\ . & & \\ . & & \\ y_1^{(n-1)}(x)\ldots\ldots.. & & y_n^{(n-1)}(x)\end{bmatrix}\cdot\begin{bmatrix}C_1'(x)\\ C_2'(x)\\ .\\ .\\ .\\ C_n'(x)\end{bmatrix}=\begin{bmatrix}0\\ 0\\ .\\ .\\ .\\ q(x)\end{bmatrix} \quad . \qquad (3.96)$$

Eqn. (3.96) has to be solved with respect to unknown functions $C_1^{'}(t), C_2^{'}(t), ..., C_n^{'}(t)$. Next by integration we are able to find functions $C_1(t), C_2(t), ..., C_n(t)$ and then a particular solution to the equation (3.1) is obtained in the form

$$y_p(t) = c_1(t)y_1(t) + c_2(t)y_2(t) + c_n(t)y_n(t) \tag{3.97}$$

Example 3.9.
Find the general solution of the following equation

$$y'' + y = \sin x \quad . \tag{3.98}$$

The corresponding to (3.98) homogeneous equation is identical with eqn. (3.80). Therefore the general solution to the homogeneous equation is

$$y_0 = C_1 \sin x + C_2 \cos x \qquad . \tag{3.99}$$

According to the above consideration it is necessary to solved the following system of equations

$$\begin{aligned} &C_1'(x)\sin x + C_2'(x)\cos x = 0 \\ &C_1'(x)\cos x - C_2'(x)\sin x = \sin x \end{aligned} \tag{3.100}$$

With respect to unknown functions $C_1^{'}(x)$ and $C_2^{'}(x)$. Dividing the first eqn. in (3.100) by sin*x* and the second one by cos *x* one gets

$$\begin{cases} C_1'(x)\sin^2 x + C_2'(x)\cos x \sin x = 0 \\ C_1'(x)\cos^2 x - C_2'(x)\sin x \cos x = \sin x \cos x \end{cases} \tag{3.101}$$

By summing up left – and right-hand sides of equations in (3.101) the following equation is obtained

$$C_1'(x) = \sin x \cos x \qquad . \tag{3.102}$$

Next integrating with respect to *x* the function $C_1(x)$ is received

$$C_1(x) = \frac{1}{2}\sin^2 x + A \quad . \tag{3.103}$$

The first equation in (3.101) gives

$$C_2'(x)\cos x \sin x = -C_1'(x)\sin^2 x \qquad . \tag{3.104}$$

Substituting (3.102) to (3.104) one obtains

$$C_2'(x)\cos x\sin x = -\sin x\cos x\sin^2 x \quad \Rightarrow \quad C_2'(x) = -\sin^2 x\,. \qquad (3.105)$$

By integrating of the last equation one gets

$$C_2(x) = -\frac{x}{2} + \frac{1}{4}\sin 2x + B\,. \qquad (3.106)$$

Assume that $A = B = 0$. Then due to eqn. (3.86) a particular solution to (3.98) is given by

$$y_p = \frac{1}{2}\sin^3 x - \frac{x}{2}\cos x + \frac{1}{2}\sin x\cos^2 x = \frac{1}{2}\sin x - \frac{x}{2}\cos x\,. \qquad (3.107)$$

According to the theorem 3.6 the general solution can be obtained by summing up the solution y_0 given by (3.99) and particular solution y_p given by (3.107). Therefore the general solution to equation (3.98) is

$$y = C_1\sin x + C_2\cos x + \frac{1}{2}\sin x - \frac{x}{2}\cos x\,. \qquad (3.108)$$

Example 3.10.
Find the general solution to the equation

$$y'' + 9y = x\,. \qquad (3.109)$$

It is easy to see that the fundamental set of solutions of corresponding to eqn. (3.109) homogeneous equation consists of functions $y_1 = \cos 3x$, $y_2 = \sin 3x$. Therefore, by applying the constant variation method, we seek for the solution in the form:

$$y = C_1(x)\cos 3x + C_2(x)\sin 3x\,. \qquad (3.110)$$

Due to eqn. (3.96) the derivatives of unknown functions can be found by solving the following set of equation

$$\begin{bmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{bmatrix} \begin{bmatrix} C_1'(x) \\ C_2'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ x \end{bmatrix}. \qquad (3.111)$$

Then by eqn. (3.94)

$$C_1'(x) = \frac{\det\begin{vmatrix} 0 & \sin 3x \\ x & 3\cos 3x \end{vmatrix}}{3\cos^2 3x + 3\sin^2 3x} = -\frac{1}{3}x\sin 3x \quad \text{and} \quad C_2'(x) = \frac{\begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & x \end{vmatrix}}{3\cos^2 3x + 3\sin^2 3x} = \frac{1}{3}x\cos 3x\,. \qquad (3.112)$$

In order to find $C_1(x)$ and $C_2(x)$ both function in eqs. (3.112) must be integrated by parts, namely

$$C_1(x) = -\frac{1}{3}\int x\sin 3x dx = \frac{1}{3}x\cdot\frac{1}{3}\cos 3x - \frac{1}{9}\int \cos 3x dx = \frac{1}{9}x\cos 3x - \frac{1}{27}\sin 3x + A. \qquad (3.113)$$

$$C_2(x) = \frac{1}{3}\int x\cos 3x d = \frac{1}{3}x\cdot\frac{1}{3}\sin 3x - \frac{1}{9}\int \sin 3x dx = \frac{1}{9}x\sin 3x + \frac{1}{27}\cos 3x + B, \qquad (3.114)$$

where A and B are integration constant. Assuming that A = B = 0, the particular solution (3.110) can be written as follows

$$\begin{aligned} y &= \frac{1}{9}x\sin 3x\sin 3x + \frac{1}{27}\cos 3x\sin 3x + \frac{1}{9}x\cos 3x\cos 3x - \frac{1}{27}\sin 3x\cos 3x = \\ &= \frac{1}{9}x(\sin^2 3x + \cos^2 3x) = \frac{1}{9}x \end{aligned}. \qquad (3.115)$$

Therefore due to Theorem 3.6 the general solution to eqn. (3.109) is

$$y = C_1\sin 3x + C_2\cos 3x + \frac{1}{9}x \quad , \qquad (3.116)$$

where C_1 and C_2 are real constant.

Theorem 3.6 shows that like in the case of the equations of the first order the method of lucky guess van be applied. In order to predict a particular solution of a given linear non-homogeneous equation the method of undetermined coefficients can be found. In the literature some theorems treating this case are formulated. These theorems, however are not discussed within this lecture note.

3.3. Applications to structural mechanics. Beams resting on Winkler-type elastic subsoil

Consider a beam as in the Figure 3.1. The beam is subjected to external load $q_0(x)$. The

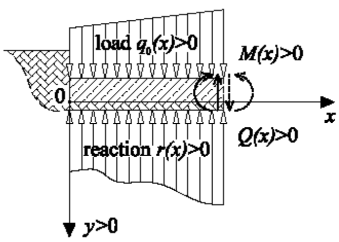

Figure 3.1 *Scheme of a beam resting on Winkler-type subsoil*

subsoil reaction is $r(x)$. We assume that the vertical beam deflection $y(x)$ is identical with vertical deformation of the subsoil. Statics gives us basic equations of Euler's type of beam. It is well-known, that the bending moment at the cross-section x satisfies the following equation:

$$M(x) = -EI\frac{d^2y(x)}{dx^2} \tag{3.117}$$

Due to Schwedler's theorem the shearing force at the cross-section x is the first derivative of bending moment, namely

$$Q(x) = \frac{dM}{dx} \quad . \tag{3.118}$$

Finally the resultant loading $q(x)$ is the first derivative of the shear force, therefore the deflection $y(x)$ has to satisfy the following equation:

$$EI\frac{d^2y(x)}{dx^2} = -q(x) \quad . \tag{3.119}$$

Let us now apply the Winkler assumption, which states that subsoil reaction is proportional to the displacement $y(x)$

$$r(x) = CBy(x) \ , \tag{3.120}$$

Where B is the width of the beam and C is the coefficient of subsoil stiffness (sometimes called the Winkler's constant). As the resultant loading is

$$q(x) = r(x) - q_0(x) = CBy(x) - q_0(x), \tag{3.121}$$

then eqn. (3.119) can be written as

$$EI\frac{d^4y(x)}{dx^4} = q_0(x) - CBy(x) \ . \tag{3.122}$$

Let us now change the coordinate system by the transformation

$$\xi = \frac{x}{L_w}, \tag{3.123}$$

where

$$L_w = \sqrt[4]{\frac{4EI}{BC}} \ . \tag{3.124}$$

The constant L_w is dimensioned in [m], therefore ξ is considered as dimensionless coordinate. By means of the chain rule one gets

$$\frac{dy}{d\xi} = \frac{dy}{dx}\frac{dx}{d\xi} = L_w\frac{dy}{dx} \quad , \tag{3.125}$$

and by successive differentiation one obtain

$$\frac{d^4 y}{d\xi^4} = L_w^4 \frac{d^4 y}{dx^4}. \tag{3.126}$$

Therefore in new coordinate system eqn. (3.122) takes the form

$$EI \frac{1}{L_w^4} \frac{d^4 y(\xi)}{d\xi^4} = q_0(\xi) - CBy(\xi) \quad . \tag{3.127}$$

After substitution (3.124) to (3.127) and some simple transformations the following equation results

$$\frac{d^4 y(\xi)}{d\xi^4} + 4y(\xi) = \frac{4q_0(\xi)}{BC} \quad . \tag{3.128}$$

Eqn. (3.128) is a non-homogeneous linear differential equation of the fourth order. First let us solve the corresponding homogeneous equation. Its characteristic equation

$$\lambda^4 + 4 = 0 \tag{3.129}$$

is equivalent to

$$\lambda^2 = 2i \qquad \lambda^2 = -2i \quad , \tag{3.130}$$

The complex roots corresponding to the first of equation in (3.130) are

$$\lambda_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 1 + i \quad \text{and} \quad \lambda_2 = \sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -1 - i \quad , \tag{3.131}$$

and the roots corresponding to the second equation in (3.131) are as follows

$$\lambda_3 = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -1 + i \quad \text{and} \quad \lambda_4 = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = 1 - i \,. \tag{3.132}$$

In virtue of the theorem 3.5 the fundamental set of solutions to homogeneous equation corresponding to eqn. (3.128) consists of functions

$$y_1 = e^{\xi}\cos\xi \,, \quad y_2 = e^{\xi}\sin\xi \,, \quad y_3 = e^{-\xi}\cos\xi \,, \quad y_4 = e^{-\xi}\sin\xi \tag{3.133}$$

and therefore the general solution is

$$y = C_1 e^{\xi}\cos\xi + C_2 e^{\xi}\sin\xi + C_3 e^{-\xi}\cos\xi + C_4 e^{-\xi}\sin\xi \,, \tag{3.134}$$

where C_1, C_2, C_3 and C_4 are integration constant. From engineering point of view more important, however, are the cases with non-zero the right hand side of the equation (3.128), namely a non-homogeneous case. Consider now the simplest case, that is the right hand side of (3.128) is independent of ξ , $q_0(\xi) = q_0$. This assumption physically means that the loading of the beam is constant. In this case it is easy to see that the function

$$y_p = \frac{q_0}{BC} \tag{3.135}$$

is a particular solution of the equation (3.128). Hence, by the theorem 3.6, the general solution to (3.129) under assumption $q_0(\xi) = q_0$ is

$$y = C_1 e^{\xi} \cos\xi + C_2 e^{\xi} \sin\xi + C_3 e^{-\xi} \cos\xi + C_4 e^{-\xi} \sin\xi + \frac{q_0}{BC} \,. \tag{3.136}$$

This way we have a complete information from mathematical point of view. For engineers, however eqn. (3.136) is useless unless the constants C_1, C_2, C_3 and C_4 remain undetermined. In order to assign specific values to these constants certain boundary conditions have to be imposed. As an example a reasonable assumption from engineering point of view is to set that bending moments and shearing forces at both sides of the beam are equal to zero, namely

$$M(0) = 0\,, \quad Q(0) = 0\,, \quad M\left(\frac{L}{L_w}\right) = 0\,, \quad Q\left(\frac{L}{L_w}\right) = 0\,, \tag{3.137}$$

Where L is the length of the beam. Application of conditions (3.137) to eqn. (3.136) leads to system of four linear algebraic equations with four unknown parameters C_1, C_2, C_3 and C_4. Solving this system (conventional solution of this system is omitted here) shows that under conditions (3.137) all four constants must be equal to zero, $C_1 = C_2 = C_3 = C_4 =0$. Therefore solution to the boundary value problem given by eqn. (3.136) and eqn. (3.137) is

$$y = \frac{q_0}{BC} \quad , \tag{3.138}$$

that is a constant function, independent of coordinate ξ. As in the Winkler's model the subsoil reaction is proportional to the displacement, then substituting eqn. (3.138) into eqn. (3.120) one gets

$$r(x) = q_0 = const. \tag{3.139}$$

The above equation show an important feature of the Winkler model. Namely, the reaction of subsoil on uniform loading of the beam is constant and equal to the intensity of the beam.

Let us now turn another important application the above model in foundation engineering. Consider a infinite beam loaded by a single force P concentrated at given point x_0 as indicated in Fig. 3.2. Then the right side in eqn. (3.128) is a non-continuous function.

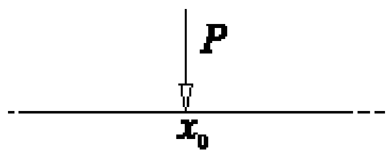

Figure 3.2. *Infinite beam subjected to a single force P*

As the scheme is symmetric with respect to axis perpendicular to beam at the point x_0. Therefore we reduce the scheme to half-infinite beam loaded at its left hand end, $\xi = 0$ (the dependence between coordinates x and ξ is given by eqn. (3.123)), as it is indicated in Fig. 3.3. Moreover, let us confine ourselves to interval $\xi > 0$, obtaining values at the point $\xi = 0$ as limits, assuming continuity of the solutions (Fig. 3.3).

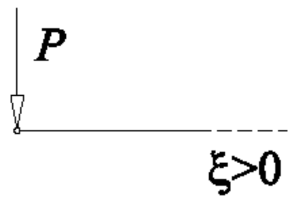

Figure 3.3. *Reduction of the problem to the half-infinite part*

In such case for $\xi > 0$ the right hand side in eqn. (3.128) vanishes and the resulting equation is

$$\frac{d^4 y(\xi)}{d\xi^4} + 4y(\xi) = 0 . \qquad (3.140)$$

The above equation will be considered together with the following boundary conditions

$$\lim_{\xi\to\infty} y(\xi) = 0 ; \quad \lim_{\xi\to 0^+} \frac{dy}{d\xi} = 0 ; \quad \lim_{x\to 0^+} Q(\xi) = -\frac{P}{2} \qquad (3.141)$$

The third condition deals with shearing force given by (3.118) and is usually called anti-symmetry condition. Equation (3.140) is the homogeneous equation corresponding to eqn. (3.128) hence the general solution to (3.140) is given by eqn. (3.134). Analysing the form of (3.134) and taking into account bounded character of trigonometric functions it is easy to see that the first and the second terms tend to infinity as ξ tends to infinity, while the third and the fourth terms tend to zero when ξ tends to infinity. Therefore to fulfill the first condition in (3.141) we have to impose that $C_1 = C_2 = 0$ and the solution (3.118) take the form

$$y = C_3 e^{-\xi} \cos\xi + C_4 e^{-\xi} \sin\xi \quad . \qquad (3.142)$$

In order to implement the second condition of (3.141) note that

$$\frac{dy}{d\xi} = (-1)\left(C_3 e^{-\xi} \cos\xi + C_4 e^{-\xi} \sin\xi\right) + e^{-\xi}\left(C_3(-\sin\xi) + C_4 \cos\xi\right) \quad . \qquad (3.143)$$

Therefore

$$\lim_{\xi\to 0^+} \frac{dy}{d\xi} = 0 \quad \Rightarrow \quad \lim_{\xi\to 0^+} \frac{dy}{d\xi} = -\left(C_3 \cdot 1 + C_4 \cdot 0\right) + 1 \cdot \left(-C_3 \cdot 0 + C_4 \cdot 1\right) = -C_3 + C_4 = 0 \Rightarrow C_3 = C_4$$

(3.144)

Consequently eqn. (3.143) takes the form

$$y = C_0 e^{-\xi}(\cos\xi + \sin\xi) \quad . \qquad (3.145)$$

In order to utilise the third condition of (3.141) we have to observe that

$$M(x) = -EI\frac{d^2y}{dx^2} \quad and \quad Q = \frac{dM}{dx} = -\frac{EI}{L_w^3}\frac{d^3y}{d\xi^3}. \qquad (3.146)$$

It easy to check that the second derivative of *y* with respect ξ is given by

$$\frac{d^2y}{d\xi^2} = 2C_0 e^{-\xi}(\sin\xi - \cos\xi) \qquad (3.147)$$

and hence

$$\frac{d^3y}{d\xi^3} = 4C_0 e^{-\xi}\cos\xi. \qquad (3.148)$$

If $\xi \to 0^+$ then $\frac{d^3y}{d\xi^3} \to 4C_0$ and therefore the third condition of (3.141) is equivalent to

$$-\frac{P}{2} = -\frac{EI}{L_w^3}4C_0 \quad \Rightarrow \quad C_0 = \frac{P}{2BCL_w} \qquad (3.149)$$

Finally, due eqn. (3.145) and eqn. (3.141), the displacements of the beam under consideration are given by

$$y(x) = \frac{P}{2BCL_w}e^{-\frac{x}{L_w}}\left(\cos\frac{x}{L_w} + \sin\frac{x}{L_w}\right) \quad for \quad x > 0. \qquad (3.150)$$

By finding second and third derivatives bending moments and shearing forces in a given cross-section of the beam can be found, which is an important task when a foundation beam is designed. In practical cases usually several different forces acting on the beam have to be considered. But such a case can be easily handled by means of the superposition theorem.

Chapter 4
Introduction to partial differential equations

4.1. Example of physical problems leading to partial differential equations and boundary value problems. The vibrating string

A tightly stretched string, whose position of equilibrium is an interval [a,b] on the *x* axis, is vibrating in the xy plane (see Fig. 4.1).

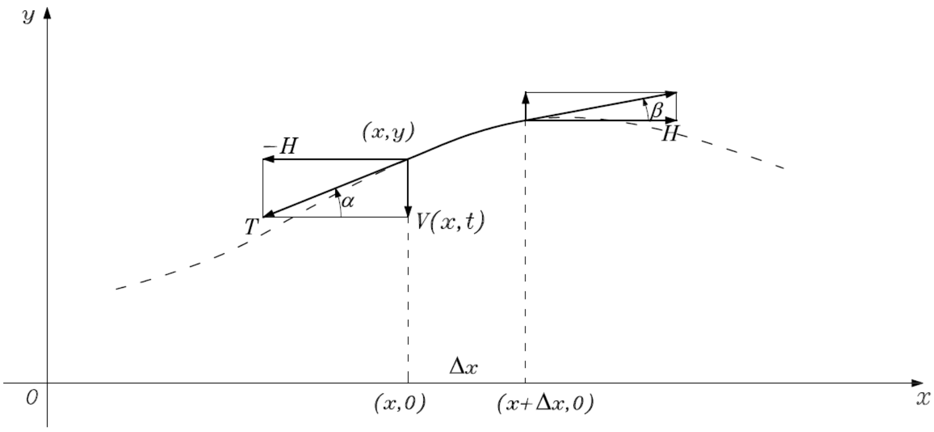

Figure 4.1. *Scheme of forces acting on a string*

In the equilibrium position each point of the string has coordinates $(x,0)$. At the time t each point is a subject of a transverse displacement $y(x,t)$ at time t. We simplify movements of the string such that each point moves in the direction of the y axis only. Then at time t the point has coordinates (x,y). Let T denotes the tension of the string. At each point (x,y) of the string the part of the string on the left of that point exerts a force of magnitude T in the tangential direction upon the part on the right . Let us denote by H the x-component of the force T. We assume that the variation of H with x and t can be neglected. The assumptions imposed above seem to be quite stringent. But it is appeared that they are quite well satisfied by strings of musical instruments under ordinary conditions of operations.

Now let $V(x,t)$ denote the y component of the tensile force exerted by the left-hand portion of the string on the right-hand portion at the point (x,y). We take the positive sense of V as that of the y axis. If a is the slope angle of the string at the point (x,y) at time t, then

$$\frac{-V(x,t)}{H} = \tan\alpha = y_x(x,t) \tag{4.1}$$

as indicated in Fig. 4.1. Thus the y component $V(x,t)$ of the force exerted at time t by the part of the string on the left of a point (x,y) upon the part on the right is given by the equation

$$V(x,t) = -Hy_x(x,t) \quad , \qquad H > 0. \tag{4.2}$$

Suppose that all external forces such as the weight of the string and resistance forces, other than forces at the end points, can be neglected. Consider a segment of the string not containing an end point and whose projection on the *x* axis has length Δx. Since x components of displacements are negligible, the mass of the segment is $\delta \Delta x$ where the constant δ is the mass of the string per unit length (mass density). At time *t* the *y* component of the force exerted by the string on the segment at the left-hand end (x,y) is $V(x,t)$, given by equation (4.2). The tangential force exerted on the other end of the segment by the part of the string on the right of that end is also indicated in Fig. 4.1. Its *y* component is given by

$$H \tan \beta = H y_X (x + \Delta x, t) \quad , \tag{4.3}$$

where β is the slope angle of the string at the right-hand end of the segment. The negative sign signifies that the force is exerted upon the part of the string on the left by the part on the right. The acceleration of the end (x,y) in the *y* direction is $y_{tt}(x,t)$. From Newton's second law of motion, it follows that

$$\delta \Delta x y_{tt}(x,t) = -H y_X(x,t) + H y_X(x + \Delta x, t) \tag{4.4}$$

approximately, when Δx is small. Hence

$$y_{tt}(x,t) = \frac{H}{\delta} \lim_{\Delta x \to 0} \frac{y_x(x + \Delta x, t) - y_x(x,t)}{\Delta x} = \frac{H}{\delta} y_{xx}(x,t) \tag{4.5}$$

at each point where the partial derivative exists. Substitution $a^2 = \frac{H}{\delta}$ leads to the equation

$$y_{tt}(x,t) = a^2 y_{XX}(x,t) \tag{4.6}$$

The constant *a* has a physical dimension of velocity. This is so-called the equation of vibrating string or one-dimensional wave equation, which is classified as a linear partial differential equation of the second order. Using the classical Leibnitz's notation it can be written as:

$$\frac{\partial^2 y(x,t)}{\partial t^2} = a^2 \frac{\partial^2 y(x,t)}{\partial x^2} \tag{4.7}$$

When external forces parallel to the *y* axis act along the string, let *F* denote the force per unit length of string, the positive sense of *F* being that of the *y* axis. Then a term $F\Delta x$ must be added on the right-hand side of equation (4.4) and the equation of motion is

$$y_{tt}(x,t) = a^2 y_{xx}(x,t) + \frac{F}{\delta} \tag{4.8}$$

In particular, with the *y* axis vertical and its positive sense upward, suppose that the external force consists of the weight of the string. Then $F\Delta x = - \delta \Delta x g$, where *g* is the acceleration of gravity; and equation (4.8) becomes the linear nonhomogeneous equation.

$$y_{tt}(x,t) = a^2 y_{xx}(x,t) - g \tag{4.9}$$

In equation (1), *F* may be a function of *x*, *t*, *y*, or derivatives of *y*. If the external force per unit length is a damping force proportional to the velocity in the *y* direction, for example, *F* is replaced by $-By_t$, where the positive constant *B* is a damping coefficient. Then the equation of motion is linear and homogeneous:

$$y_{tt}(x,t) = a^2 y_{xx}(x,t) - b y_t(x,t) \quad , \quad b = \frac{B}{\delta} \tag{4.10}$$

If one end $x = 0$ of the string is kept fixed at the origin at all times $t \ge 0$, the boundary condition there is clearly

$$y(0,t) = 0 \quad , \quad t \ge 0 \tag{4.11}$$

But if that end is permitted to slide along the *y* axis and if the end is moved along that axis with a displacement *f*(*t*), the boundary condition is the linear nonhomogeneous condition

$$y(0,t) = f(t) \quad , \quad t \ge 0 \tag{4.12}$$

When the left-hand end is looped around the *y* axis and a force *g*(*t*), ($t > 0$) in the *y* direction is applied to that end, *g*(*t*) is the limit of the force *V*(*x*,*t*) described above as *x* tends to zero through positive values. The boundary condition is then

$$-Hy_x(0,t) = g(t) \quad , \quad t > 0 \quad . \tag{4.13}$$

The negative sign disappears, however, if $x = 0$ is the right-hand and because *g*(*t*) is then the force exerted on the part of the string to the left of that end.

4.2. Notations and preliminary definitions

Classical notation. According to the classical notation, which has been proposed by Leibnitz, the partial derivative of order *k* of function u with respect independent variables x_i and x_j is denoted by

$\dfrac{\partial^k u(\mathbf{x})}{\partial x_j^l \delta_i^{k-l}}$, for any integer $l < k$.

Lower index notation. In this kind of notations the partial derivative with respect independent variable *x* or *t* is denoted by y_x or y_t . Second derivatives are denoted by y_{xx}, y_{xt}, y_{tt}.

General index notation

Let $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$ be a vector with non-negative components. The vector $\boldsymbol{\alpha}$ is called a *multiindex* with the *length* equal to $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$. If $\boldsymbol{\alpha}$ is a given multiindex, then $\mathrm{D}^{\alpha} u(\mathbf{x})$ denotes the following partial derivative of the function *u* of order $|\alpha|$:

$$\mathrm{D}^{\alpha} u(\mathbf{x}) = \frac{\partial^{|\alpha|} u(\mathbf{x})}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} ... \partial_{x_n}^{\alpha_n} u \quad . \tag{4.14}$$

If k is a positive integer then

$$\mathrm{D}^k u(x) = \{\mathrm{D}^\alpha u(x) : |\alpha| = k\} \tag{4.15}$$

constitutes the set of all partial derivatives of order k. If $k = 1$ one gets

$$\mathrm{D}u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, ..., \frac{\partial u}{\partial x_n}\right) = \mathrm{grad}\, u \quad , \tag{4.16}$$

for a function u of the class $C^1((-\infty, +\infty))$, the *gradient* of the function u. In the case of $k = 2$, if u is a function of $C^2((-\infty, +\infty))$, elements of $\mathrm{D}^2 u$ are elements of matrix

$$\mathrm{D}^2 u = \begin{pmatrix} \frac{\partial^2 u}{\partial x_1^2} \cdots \frac{\partial^2 u}{\partial x_1 \partial x_n} \\ \cdots\cdots\cdots\cdots \\ \frac{\partial^2 u}{\partial x_n \partial x_1} \cdots \frac{\partial^2 u}{\partial x_n^2} \end{pmatrix} \quad , \tag{4.17}$$

Which is known as a *hesjan*. The trace of the matrix (4.17):

$$\Delta u = tr(\mathrm{D}^2 u) = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2} \tag{4.18}$$

is the *laplasjan* of the function u.

After presentation of notations that can be met in the course of partial differential equations the definition of the partial differential equation can be formulated.

Definition 4.1. Let U be an open set in $\mathbf{R^n}$. Consider an equation of the form

$$F\left(\mathrm{D}^k u(\mathbf{x}), \mathrm{D}^{k-1} u(\mathbf{x}), ..., \mathrm{D}u(\mathbf{x}), u(\mathbf{x}), \mathbf{x}\right) = 0, \tag{4.19}$$

where $\mathbf{x} \in U \subset \mathbf{R^n}$, $F : \mathbf{R^{n^k}} \times \mathbf{R^{n^{k-1}}} \times \mathbf{R^n} \times \mathbf{R} \times U \to \mathbf{R}$ is a given function and $u : U \to \mathbf{R}$ is the unknown function. Equation (4.19) is called a partial differential equation of the order k.

4.3. Classification

Among partial differential equations the special role is played by linear differential equations. This is because historically almost all problems in physics and mechanics had led to this type of equations. On the other hand a closed form solutions for an equation which is not linear can be hardly obtain. Below definitions of linear equations together with some relative forms are given.

Definition 4.2. A partial differential equation of order k is called linear if it can be expressed in the following form

$$\sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u = f(x) \quad , \tag{4.20}$$

where $|\alpha| \le k$ and $a_\alpha(x)$ and $f(x)$ are given functions. Functions $a_\alpha(x)$ do not depend on neither any derivative of the function u nor the function u itself. If $f \equiv 0$ then equation (4.20) is called *homogeneous*.

Definition 4.3. A partial differential equation of order k is called half-linear or semi-linear if it can be expressed in the following form

$$\sum_{|\alpha|=k} a_\alpha(x) D^\alpha u + a_\alpha(D^{k-1} u,..., Du, u, x) = 0 \qquad (4.21)$$

Definition 4.4. A partial differential equation of order k is called quasi-linear if it can be expressed in the following form

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u,..., Du, u, x) D^\alpha u + a_\alpha(D^{k-1} u,..., Du, u, x) = 0 \quad . \qquad (4.22)$$

Definition 4.5. A partial differential equation (4.19) of order k is called totally non-linear if function F depends on derivatives of order k in a non-linear way.

Definition 4.6. An equation of the form

$$\overline{F}(D^k \overline{u}(x), D^{k-1}\overline{u}(x),..., D\overline{u}(x), \overline{u}(x), x) = \overline{0} \quad , \qquad (4.23)$$

where $\overline{F}$ is a given known function of the type $\overline{F} : R^{mn^k} \times \Re^{nn^{k-1}} \times .... \times R^{mn} \times \Re^m \times U \to R^m$, $\overline{0}$ is the column vector of zeros and $\overline{u} : U \to \Re^m, \overline{u} = (u^1,..u^m)$ is and unknown function is called a system of partial differential equations of order k.

4.4. Some important examples of partial differential equations

In this section some most well-known in physics partial differential equations are listed.

Linear equations

The Laplace equation

$$\Delta u = \sum_{i=1}^{n} u_{x_i x_i} = 0 \qquad (4.24)$$

The Helmholz equation (eigenvalue problem of the Laplace operator)

$$-\Delta u = \lambda u \quad , \quad \lambda \in \mathbf{R} \qquad (4.25)$$

The linear transport equation

$$u_t + \sum_{i=1}^{n} b^i u_{x_i} = 0 \qquad (4.26)$$

The Liouville equation

$$u_t - \sum_{i=1}^{n} b^i u_{x_i} = 0 \qquad (4.27)$$

The heat conduction equation

$$u_t - \Delta u = 0 \qquad (4.28)$$

The Schrödinger equation

$$i\frac{h}{2\pi}u_t + \frac{h^2}{8\pi m}\Delta u = 0 \qquad (4.29)$$

The Kolmogorov equation

$$u_t - \sum_{i,j=1}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} = 0 \qquad (4.30)$$

The Fokker-Planck equation

$$u_t - \sum_{i,j=1}^{n} (a^{ij}u)_{w_i x_j} - \sum_{i=1}^{n} (b^i u)_{x_i} = 0 \qquad (4.31)$$

The wave equation

$$u_{tt} - \Delta u = 0 \qquad (4.32)$$

The general wave equation

$$u_{tt} - \sum_{i,j}^{n} a^{ij} u_{x_i x_j} + \sum_{i=1}^{n} b^i u_{x_i} = 0 \qquad (4.33)$$

The Airy equation

$$u_t + u_{xxx} = 0 \qquad (4.34)$$

Half-linear equations

The non-linear Poisson equation

$$-\Delta u = f(u) \qquad (4.35)$$

The non-linear wave equation

$$u_{tt} - \Delta u = f(u) \qquad (4.36)$$

Totally non-linear equations

The Hamilton-Jacoby equation

$$u_t + H(Du, x) = 0 \qquad (4.37)$$

The scalar law of conservation

$$u_t + div\overline{F}(u) = 0 \quad . \qquad (4.38)$$

4.5. The classical classification of the linear equations of the second order

Consider a domain include in $\mathbf{R^2}$. The second-order linear partial differential equation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + Fu = G \quad , \qquad (4.39)$$

Where *A, B,…, G* are constants of functions of *x* and *y* only, is ***elliptic***, ***parabolic***, or ***hyperbolic*** type in a domain of the *xy* pane if the quantity

$$B^2 - 4AC \qquad (4.40)$$

is negative, zero, or positive, respectively, throughout the domain. The three types require different kinds of boundary conditions to determine a solution.

Example 4.1. Consider the Laplace equation in the form

$$\frac{\partial^2 u}{\partial x^x} + \frac{\partial^2 u}{\partial y^2} = 0 \quad . \qquad (4.41)$$

For this case *A* = *C* = 1 and *B, D, E, F* and *G* vanish. Therefore $B^2 - 4AC = -4 < 0$, hence the Laplace equation is elliptic in every domain.

Example 4.2. In the case of wave equation (4.7), $B^2 - 4AC = +4a^2 > 0$, hence the wave equation is hyperbolic in every domain.

Example 4.3. Consider the heat conduction equation of the form

$$k\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y} = 0 \qquad (4.42)$$

For this case *A* = *k, E* = 1 and *B, C, D, F* and *G* vanish. Therefore $B^2 - 4AC = 0$, hence the heat conduction equation is parabolic in every domain.

4.6. Boundary value problems

Example 4.3. The problem consisting of the partial differential equation (the Laplace equation)

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0 \qquad \text{for } x \rangle 0 \text{ and } y \rangle 0 \qquad (4.43)$$

and the two boundary conditions

$$u(0,y) = \frac{\partial u(0,y)}{\partial x} \quad , \qquad y > 0 \qquad (4.48)$$

$$u(x,0) = \sin x + \cos x \quad , \qquad x > 0 \qquad (4.49)$$

is an example of a boundary value problem in partial differential equations. The domain is the first quadrant of the *xy* plane. Let us verify that the function

$$u(x,y) = e^{-y}(\sin x + \cos x) \qquad (4.50)$$

is a solution of that problem. First let us find partial derivatives of the function given by eqn. (4.50).

$$\frac{\partial u}{\partial y} = -e^{-y}(\sin x + \cos x) \quad \Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = e^{-y}(\sin x + \cos x) \qquad (4.51)$$

$$\frac{\partial u}{\partial x} = e^{-y}(\cos x - \sin x) \quad \Rightarrow \quad \frac{\partial^2 u}{\partial x^2} = (-1)e^{-y}(\sin x + \cos x) \qquad (4.52)$$

The above equations evidently show that function *u* defined by (4.50) satisfies the Laplace equation. Moreover

$$u(0,y) = e^{-y} \quad \textit{and} \quad \frac{\partial u}{\partial x}(0,y) = e^{-y}(\cos 0 - \sin 0) = e^{-y} \quad \Rightarrow \quad u(0,y) = \frac{\partial u}{\partial x}(0,y) \qquad (4.53)$$

and

$$u(x,0) = e^{0}(\sin x + \cos x) = \sin x + \cos x \,. \qquad (4.54)$$

This means that the boundary conditions are fulfilled by the function (4.50).

Let *u* denote the unknown function in a boundary value problem. A condition that prescribes the values of *u* itself along a boundary is known as a boundary condition of the ***first type***, or a ***Dirichlet condition***. A boundary condition of the ***second type***, also called a ***Neumann condition***, proscribes the values of the normal derivative *du/dn* of the function at the boundaries. Among other kinds of boundary conditions are those of the ***third type*** in which values of *hu + du/dn* are prescribed at the boundaries, where *h* is either a constant or a function of the independent variables.

If the partial differential equation in *u* is of second order with respect to one of the independent variables *t* and if the values of both *u* and u_t are prescribed on a boundary $t = 0$, the boundary condition is one of ***Cauchy type*** with respect *t*.

From the viewpoint of applications it is usually very important to avoid of so-called ill-conditioned problems. The notion is explained by the definition given below.

Definitin 4.7. **(Hadamard's well-posedness)**

A boundary value problem is said to be well-posed if

1. it has a solution,
2. the solution is unique,
3. the solution depends continuously on the given data.
Otherwise the problem is ill-posed or ill-conditioned.

Chapter 5
Some simple methods of solving partial differential equations and boundary value problems

5.1. Successive integration

As in the case of ordinary differential equations as one of the most important notions are particular and general solutions. The definitions are given below.

Definition 5.1. A particular solution of a given partial differential equation of order *n* in the domain *D* is called a function of the class $C^n(D)$, which satisfies the given equation in each point of *D*.

Example 5.1. Consider the equation

$$xu_x - yu_y = u \quad . \tag{5.1}$$

Let us demonstrate that the function

$$u(x, y) = x^2 y \tag{5.2}$$

Is the particular solution to eqn. (5.1) in any domain $D \subset \mathbf{R^2}$. It is obvious that the function defined by (5.2) is of the class $C^1(D)$. Substitution of the function *u* into the left-hand side of the eqn. (5.1) gives

$$x\frac{\partial}{\partial x}\left(x^2 y\right) - y\frac{\partial}{\partial y}\left(x^2 y\right) = x2xy - yx^2 = yx^2 \quad . \tag{5.3}$$

This means that the equation (5.1) is satisfied independently of the choice of the domain *D*.

Definition 5.2. A general solution to a given partial differential equation of order *n* is the family of its all particular solutions.

In order to illustrate similarities and differences between ordinary and partial differential equations, let us start with some very simple examples.

Example 5.1. Find a general solution to the following equation of the second order:

$$\frac{\partial^2 u}{\partial x \partial y} = 0 \tag{5.4}$$

The equation (5.4) can be written as

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 0 \, , \tag{5.5}$$

which means that the derivative $\frac{\partial u}{\partial y}$ is constant with respect to the variable *x*. Then it follows that

$$\frac{\partial u}{\partial y} = g(y) \quad , \tag{5.6}$$

where *g* is any function of class $C^1((-\infty, +\infty))$. Finally function *u* can be obtained by integrating the above derivative with respect to variable *y* , namely

$$u = \int g(y)dy + \psi(x), \tag{5.7}$$

where $\psi(x)$ is any function of the class $C^2((-\infty, +\infty))$ of variable *x* solely. If now $\varphi(y)$ is any antiderivative of the function *g*, then it follows that the general solution to the eqn. (4.11) has a form:

$$u(x, y) = \varphi(y) + \psi(x) \tag{5.8}$$

Note that the general solution (5.8) contains a huge number of functions in comparison with analogical problems in ordinary differential equations.
Example 4.2. Find the general solution to the equation

$$\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad . \tag{5.9}$$

Let us introduce new independent variables in the following way

$$\xi = x + y \quad ; \quad \eta = x - y \quad . \tag{5.10}$$

Consequently

$$u(x, y) = u\left[\frac{1}{2}(\xi + \eta), \frac{1}{2}(\xi - \eta)\right] = h(\xi, \eta) \tag{5.11}$$

and

$$\frac{\partial u}{\partial x} = \frac{\partial h}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial h}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial h}{\partial \xi} + \frac{\partial h}{\partial \eta}. \tag{5.12}$$

Similarly

$$\frac{\partial u}{\partial y} = \frac{\partial h}{\partial \xi} - \frac{\partial h}{\partial \eta} \quad . \tag{5.13}$$

Substituting (5.12) and (5.13) to (5.9) one gets

$$\frac{\partial h}{\partial \eta} = 0 \quad . \tag{5.14}$$

Integrating the last equation with respect to variable η one obtain

$$h(\xi,\eta) = g(\xi) \quad , \tag{5.15}$$

where g is any function of the class $C^1(-\infty;+\infty)$. Coming back to initial coordinate system the general solution o eqn. (5.9) appears as

$$u(x,y) = g(x+y) \quad . \tag{5.16}$$

5.2. Separation of variables

In section 2.1. ordinary differential equations of separated variables has been considered. Here separation of variables will be applied as a method of finding solutions of selected boundary value problems associated with certain types of partial differential equations. The method will be demonstrated on the basis of the string equation (4.6). Assume that $x \in [0,c]$ and $t > 0$. We will seek for a solution which satisfies the following boundary conditions:

$$y(0,t) = 0, \quad y(c,t) = 0, \quad y_t(x,0) = 0. \tag{5.17}$$

In determining nontrivial (trivial solution means that $y \equiv 0$) solutions of all homogeneous equations (4.6) and (5.17) in the above boundary value problem, using ordinary differential equations we seek functions of the form

$$y(x,t) = X(x) \cdot T(t) \tag{5.18}$$

which satisfy those equations. Note that X is a function of x alone and T a function of t alone. Note, too, that X and T must be nontrivial. If $y = XT$ satisfies equation (4.6), then

$$X(x) \cdot T''(t) = a^2 x''(x) \cdot T(t)\,; \tag{5.19}$$

And we can divide by a^2XT to separate variables

$$\frac{X''(X)}{X(X)} = \frac{T''(t)}{a^2T(t)} \qquad \forall x \in [0,c] \quad \forall_t > 0 \tag{5.20}$$

Since the left-hand side here is a function of x alone, it does not very with t. However, it is equal to a function of t alone, and so it cannot vary with x. Hence both sides must be some constant value, which we denote as $-\lambda$ in common, that is

$$X''(x) = -\lambda X(x) \qquad T''(t) = -\lambda a^2 T(t) \quad . \tag{5.21}$$

If XT is to satisfy the first of conditions (5.17), then $X(0)T(t)$ must vanish for all $t > 0$. But T is nontrivial then it follows that $X(0) = 0$. Likewise, the last two conditions of (5.17) are satisfied by XT if $X(c) = 0$ and $T'(t) = 0$. Thus XT satisfies equations (4.6) and (5.17) when X and T satisfy these two homogeneous problems:

$$X''(x) + \lambda X(x) = 0, \quad X(0) = 0, \quad X(c) = 0 \tag{5.22}$$

$$T''(t) + \lambda a^2 T(t) = 0, \qquad T'(0) = 0 \quad , \tag{5.23}$$

where the parameter λ has the same value in both problems. Note that problem (5.23) has only one boundary condition and therefore many solutions for each λ. Since problem (5.22) has two boundary conditions, it may have nontrivial solutions for particular values of λ.

If $\lambda = 0$, then

$$\lambda = 0 \quad \Rightarrow \quad X''(x) = 0 \Rightarrow \quad X'(x) = A \Rightarrow \quad X(x) = Ax + B \quad . \tag{5.24}$$

On the other hand

$$X(0) = 0 \quad \Rightarrow \quad B = 0; \quad X(c) = 0 \Rightarrow A \cdot C + 0 = 0 \quad \Rightarrow A = 0\,. \tag{5.25}$$

Therefore his problem has just trivial solution $X(x) \equiv 0$ when $\lambda = 0$.

If $\lambda > 0$, we may write $\lambda = \alpha^2$ ($\alpha > 0$). The differential equation in problem (5.22) takes the form

$$X'' + \alpha^2 X = 0 \tag{5.26}$$

Its general solution is

$$X(x) = C_1 \sin \alpha x + C_2 \cos \alpha x \tag{5.27}$$

The condition $X(0) = 0$ implies that $C_2 = 0$; and if the condition $X(c) = 0$is hold,

$$C_1 \sin \alpha c = 0 \quad . \tag{5.28}$$

In order for there to be a nontrivial solution, then, α must satisfy the equation

$$\sin \alpha c = 0 \Rightarrow \alpha = \frac{n\pi}{c} \qquad n = 1,2... \tag{5.29}$$

Thus, except for the constant factor C_1,

$$X(x) = \sin \frac{n\pi x}{c} \qquad n = 1,2... \tag{5.30}$$

The numbers $\lambda = \alpha^2 = \dfrac{n^2\pi^2}{c^2}$ for chich problem (5.22) has nontrivial solution are called *eigenvalues* of the problem, and the functions (5.30) are the corresponding *eigenfunctions*.

When $\lambda < 0$, let us write $\lambda = -\beta^2$ ($\beta > 0$). Then

$$X(x) = D_1 \sinh(\beta x) \tag{5.31}$$

is the solution of linear homogeneous equation

$$X'' - \beta^2 X = 0 \tag{5.32}$$

That satisfies the condition $X(0) = 0$. Since $\sinh(\beta c) \neq 0$ then $D_1 = 0$ if $X(c) = 0$. Thus the problem (5.22) has no negative eigenvalues.

When $\lambda = \frac{n^2\pi^2}{c^2}$, problem (5.23) becomes

$$T''(t) + \left(\frac{n\pi a}{c}\right)^2 T(t) = 0; \qquad T(0) = 0 \quad . \tag{5.33}$$

Except for a constant factor, then

$$T(t) = \cos\frac{n\pi at}{c} \quad . \tag{5.34}$$

Therefore each function of the infinite set

$$y_n(x,t) = \sin\frac{n\pi x}{c}\cos\frac{n\pi at}{c} \quad , \quad n = 1,2,..... \tag{5.35}$$

satisfies all homogeneous equations (4.6) and (5.17).

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